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SUMS OF LINKED IDEALS

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ABSTRACT. It is shown that the sum of two geometrically linked ideals in the linkage class of a complete intersection is again an ideal in the linkage class of a complete intersection. Conversely, every Gorenstein ideal (of height at least two) in the linkage class of a complete intersection can be obtained as a "generalized localization" of a sum of two geometrically linked ideals in the linkage class of a complete intersection. We also investigate sums of doubly linked Gorenstein ideals. As an application, we construct a perfect prime ideal which is strongly nonobstructed, but not strongly Cohen-Macaulay, and a perfect prime ideal which is not strongly nonobstructed, but whose entire linkage class is strongly Cohen-Macaulay.

INTRODUCTION

Let X and Y be two projective varieties of codimension g that have no common components and no embedded components. Then X and Y are *geometrically linked* if the scheme theoretic union $X \cup Y$ is a complete intersection [28]. In this paper now, we are mainly interested in the structure of the scheme theoretic intersection $X \cap Y$ of the two linked varieties X and Y.

More generally, two ideals I and J in a local Gorenstein ring R are said to be *linked* (write $I \sim J$) if there is an R-regular sequence $\alpha = \alpha_1, \dots, \alpha_g$ contained in $I \cap J$ such that $J = (\alpha) : I$ and $I = (\alpha) : J$ [28]. If moreover I and J have no minimal primes in common, then the link $I \sim J$ is called *geometric* [28]. We say that an ideal I is *licci* (in the *l*inkage *c*lass of a *c*omplete *i*ntersection) if there is a sequence of links $I \sim I_1 \sim \cdots \sim I_n$ with I_n a complete intersection (standard examples of licci ideals include perfect ideals of grade two, [2], [8], and perfect Gorenstein ideals of grade three, [35]). Peskine and Szpiro have shown that if I and J are two geometrically linked Cohen-Macaulay ideals of grade g, then $I + J$ is a Gorenstein ideal of grade $g + 1$ [28], thus providing a way to construct Gorenstein ideals from Cohen-Macaulay ideals of smaller grade. It seems natural to ask which properties are preserved under this construction, as we pass from I or J to $I + J$. We are mainly interested in the property of being licci, partly because other than linkage, only very few methods are known of obtaining new licci ideals from given ones.

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Now let R be a local Gorenstein ring with infinite residue class field. We are going to show that the sum of two geometrically linked licci R-ideals is indeed a licci (Gorenstein) ideal (Theorem 2.1). As a partial converse we prove that every licci Gorenstein R-ideal (of grade at least two) can be obtained as a generalized localization of a sum of two geometrically linked licci R-ideals (Theorem 2.17). Here "generalized localizaton" is a technical notion comprising localization, deformation, specialization, and purely transcendental changes of the residue class field (Definition 1.8). This notion cannot be deleted in the statement of Theorem 2.17, since there exist licci Gorenstein ideal which are not equal to a sum of two geometrically linked licci ideals (Proposition 2.5, Corollary 2.7, Example 2.8). Combining Theorems 2.1 and 2.17, we eventually obtain the following characterization: Let K be an R-ideal of grade at least two, then K is a licci Gorenstein ideal if and only if K is a generalized localization of the sum of two geometrically linked licci R-ideals (Corollary 2.28). To prove these results, we repeatedly modify the links in question by using the notion of generalized localization, and eventually reduce the claim to the case, where the ideals have grade one or two and the problem can be solved by explicit computation. These proofs can be found in the second section of the paper, whereas the necessary background information about generalized localization is developed in the first section.

In the third section, we deal with the question of how depth conditions on the Koszul homology and the twisted conormal module pass to the sum of geometrically linked ideals (Corollary 3.10, Theorem 3.11). To answer this question, we need another result which might be of independent interest (Theorem 3.1, Corollaries 3.4 and 3.5): Let I and J be two Cohen-Macaulay ideals which are linked in an odd number of steps, then the depth of the first Koszul homology of I equals the depth of the second symmetric power of the canonical module of R/J (whereas by Huneke, [13], the depths of the first Koszul homologies of I and J coincide, in case I and J are linked in an even number of steps).

In the fourth section, we turn our attention to the sum of ideals which are linked in two steps. Let $I_0 \sim I_1 \sim I_2$ be a double link where now I_0 and I_2 are Gorenstein R-ideals of grade g. We further assume that (R, I_0) and (R, I_2) have no common deformation (cf. Definition 1.7). In this situation, we show that $I_0 \sim I_1 \sim I_2$ is a tight double link in the sense of Kustin and Miller ([24], cf. Definition 1.5) (Proposition 2.16), and that the "generic" grade of the sum $I_0 + I_2$ is $g+2$ (Lemma 4.1, Remark 4.2). Thus it is natural to suppose that $I_0 + I_2$ has grade at least $g+2$, and with this assumption, we show that $I_0 + I_2$ is a Cohen-Macaulay ideal of grade $g+2$ provided the first Koszul homology of I_0 is Cohen-Macaulay (Theorem 4.3).

In the fifth section, we use the results of the preceding two sections to construct two examples. Recall that an ideal I is called *strongly Cohen-Macaulay* if all the Koszul homology modules of I are Cohen-Macaulay, and *strongly* *nonobstructed* if I tensored with the canonical module of R/I is a Cohen-Macaulay module (for the significance of these notions in the study of blowing-up rings, residual intersections, and deformations, cf. [5], [10], [12], [14], [16], [21], [30], [31]). Huneke has shown that the strong Cohen-Macaulayness is preserved under even linkage ([13], [15]), and Buchweitz has proved that the property of being strongly nonobstructed passes through any number of links ([5], cf. also [6]). In particular, if I is licci, then the entire linkage class of I is strongly Cohen-Macaulay and I is strongly nonobstructed. Now in the last section of this paper, we consider a sum of two doubly linked Gorenstein ideals to construct a perfect prime ideal (generated by a d-sequence) which is not strongly nonobstructed, but whose entire linkage class is strongly Cohen-Macaulay. Conversely, we essentially take a sum of two geometrically linked perfect ideals to obtain a Gorenstein prime ideal which is strongly nonobstructed, but not strongly Cohen-Macaulay (Theorem 5.3). By the results of Buchweitz and Huneke, the two ideals cannot be licci, thus demonstrating that neither the strong Cohen-Macaulayness of the entire linkage class nor the strong nonobstructedness of the ideal suffice to characterize licci ideals.

1. Linkage and generalized localization

In this section we fix the notation that will be used throughout the paper and list some general results concerning linkage. We also introduce the notion of generalized localization and derive its basic properties, which will play an important role in the next section.

In this paper, "ideal" will always mean proper ideal. Let (R, m) be a Noetherian local ring, let I be an R-ideal, and let M be a finitely generated R-module. Then $l(M)$ denotes the length of M, $\nu(M)$ is the minimal number of generators of M, M satisfies Serre's condition (S_k) if $\operatorname{depth} M_p \geq \min\{k, \dim R_p\}$ for all $p \in \operatorname{Spec}(R)$, and we say that M has a rank and that this rank is n if $M_p \cong \bigoplus^n R_p$ for all $p \in \operatorname{Ass}(R)$. By $S_n(M)$ we denote the nth symmetric power of M. The grade of I, $\operatorname{grade} I$, is the maximal length of an R-regular sequence inside I, and the deviation of I, $d(I)$, is the difference $\nu(I)$-grade I. The ideal I is called unmixed if all associated primes of I have the same height in R, and I is a complete intersection (almost complete intersection) if $d(I) = 0$ ($d(I) \leq 1$ respectively). We will often say that I is Cohen-Macaulay or Gorenstein, by which we mean that the ring R/I has any of these properties. An ideal in a local Cohen-Macaulay ring is called perfect if it is Cohen-Macaulay and has finite projective dimension. Further we define $V(I) = \{p \in \operatorname{Spec}(R) | I \subset p\}$, $NCI(I) = \{p \in V(I) | I_p$ not a complete intersection$\}$, and $NG(R) = \{p \in \operatorname{Spec}(R) | R_p$ not Gorenstein$\}$. The ideal I satisfies (CI_k) if $\dim(R/I)_p > k$ for every $p \in NCI(I)$, I is generically a complete intersection if I is unmixed and (CI_0), and the ring R satisfies Serre's condition (R_k) if R_p is regular for all $p \in \operatorname{Spec}(R)$ with $\dim R_p \leq k$. By ω_R we denote the canonical module of R (in case it exists), and $r(R)$ is the type of R (in case R is Cohen-Macaulay). Of course, $r(R) = \nu(\omega_R)$. If X is a finite set of variables, we write $R(X) = R[X]_{(m)}$. By $I_{t\times t}$ we denote the t by t identity matrix, and for a matrix A with entries in R, $I_t(A)$ is the R-ideal generated by all t by t minors of A.

Definition 1.1 [28]. Let I and J be two ideals in a (not necessarily local) Gorenstein ring R.

(a) I and J are said to be *linked* (with respect to $\boldsymbol{\alpha}$) (write $I \sim J$) if there exists an R-regular sequence $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ in $I \cap J$ such that $J = (\boldsymbol{\alpha}) : I$ and $I = (\boldsymbol{\alpha}) : J$.

(b) I and J are said to be *geometrically linked* if I and J are linked and if in addition $\mathrm{Ass}(R/I) \cap \mathrm{Ass}(R/J) = \emptyset$.

Notice that in the above definition, I and J are not allowed to be unit ideals, and that I and J are automatically unmixed ideals of the same grade g. Moreover, I and J are geometrically linked if and only if I and J are linked and $\mathrm{grade}(I + J) \geq g + 1$. In this case, $I \cap J = (\boldsymbol{\alpha})$ [28].

Proposition 1.2 [28]. *Let R be a (not necessarily local) Gorenstein ring, let I be an unmixed R-ideal of* grade g, *let $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ be a regular sequence contained in I with $(\boldsymbol{\alpha}) \neq I$, and set $J = (\boldsymbol{\alpha}) : I$. Then*

(a) *$I = (\boldsymbol{\alpha}) : J$ (i.e., I and J are linked).*
(b) *I is Cohen-Macaulay if and only if J is Cohen-Macaulay.*
(c) *Let in addition R be local and assume that I is Cohen-Macaulay. Then $\omega_{R/J} \cong I/(\boldsymbol{\alpha})$ and $\omega_{R/I} \cong J/(\boldsymbol{\alpha})$. In particular, $r(R/J) = \nu(I/(\boldsymbol{\alpha}))$ and $r(R/I) = \nu(J/(\boldsymbol{\alpha}))$.*

It is immediate from Proposition 1.2(c) that if I is Gorenstein, then J is an almost complete intersection. Conversely, if J is an almost complete intersection, but not a complete intersection, then I is Gorenstein if and only if $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ form part of a minimal generating set of J.

Proposition 1.3 [28]. *Let R be a local Gorenstein ring, let I and J be two geometrically linked Cohen-Macaulay ideals of* grade g, *and set $K = I + J$.*

Then K is a Gorenstein ideal of grade $g + 1$.

The ideal K from the above proposition is clearly the preimage in R of a canonical ideal of R/I, because

$$K/I = I + J/I \cong J/I \cap J \cong J/(\boldsymbol{\alpha}) \cong \omega_{R/I}$$

(cf. Proposition 1.2(c)). Proposition 1.3 also provides a natural way of constructing Gorenstein ideals from Cohen-Macaulay ideals of smaller grade. We are going to address the question of which Gorenstein ideals K actually arise in this way as a sum of two geometrically linked Cohen-Macaulay ideals I and J, and which properties of I or J are preserved as we pass to $K = I + J$. We are particularly interested in the property of being in the linkage class of a complete intersection:

Definition 1.4. Let I be an ideal in a local Gorenstein ring R.

(a) I is said to be in the *linkage class* of an R-ideal J if there is a sequence of links in R, $I = I_0 \sim I_1 \sim \cdots \sim I_n = J$. If n is even (*odd*) then I is in the *even* (*odd*) *linkage class* of J, and if $n = 2$ then I and J are *doubly linked*.

(b) I is called *licci* if it is in the *l*inkage *c*lass of a *c*omplete *i*ntersection ideal.

It follows from Proposition 1.2(b) that licci ideals are automatically Cohen-Macaulay. Before we proceed, several more definitions are needed.

Definition 1.5 [24]. Let R be a local Gorenstein ring, and let $I_0 \sim I_1 \sim I_2$ be a sequence of links in R, where I_0 and I_2 are Gorenstein R-ideals of grade $g > 0$. Then $I_0 \sim I_1 \sim I_2$ is called a *tight double link*, if there are elements $\beta, \delta, \alpha = \alpha_1, \ldots, \alpha_{g-1}$ in R, such that α, β is an R-regular sequence defining the link $I_0 \sim I_1$, α, δ is an R-regular sequence defining the link $I_1 \sim I_2$, and $I_1 = (\alpha, \beta, \delta)$.

Definition 1.6 [18]. Let (R, I) and (S, J) be pairs, where R, S are Noetherian rings, and $I \subset R$, $J \subset S$ are ideals or $I = R$ or $J = S$.

(a) We say that (R, I) and (S, J) are isomorphic, and write $(R, I) \cong (S, J)$, if there is an isomorphism $\phi: R \to S$ with $\phi(I) = J$.
(b) We say that (R, I) and (S, J) are *equivalent*, and write $(R, I) \equiv (S, J)$, if there are finite sets of indeterminates X over R and Z over S such that $(R[X], IR[X]) \cong (S[Z], JS[Z])$.
(c) In addition assume that R and S are local. We say that (R, I) and (S, J) are *generically equivalent*, and write $(R, I) \approx (S, J)$, if there are finite sets of indeterminates X over R and Z over S such that $(R(X), IR(X)) \cong (S(Z), JS(Z))$.

Definition 1.7 [18]. Let (R, I) and (S, J) be pairs as in Definition 1.6(c).

(a) We say that (S, J) is a *deformation* of (R, I) (or equivalently, (R, I) is a *specialization* of (S, J)) (with respect to $\mathbf{a}$), if there is a sequence $\mathbf{a} \subset S$, $\mathbf{a} = (a_1, \ldots, a_n)$, which is regular on S and S/J such that $(S/(\mathbf{a}), (J + \mathbf{a})/(\mathbf{a})) \cong (R, I)$.
(b) We say that (S, J) is *essentially a deformation* of (R, I), if there is a sequence of pairs (S_i, J_i), $1 \leq i \leq n$, with $(S_1, J_1) = (R, I)$, $(S_n, J_n) = (S, J)$, such that for each $1 \leq i \leq n-1$, one of the following conditions is satisfied:
(i) $(S_{i+1}, J_{i+1}) = ((S_i)_p, (J_i)_p)$ for some $p \in \operatorname{Spec}(S_i)$.
(ii) (S_{i+1}, J_{i+1}) is a deformation of (S_i, J_i).
(iii) $(S_{i+1}, J_{i+1}) \approx (S_i, J_i)$.

We have to introduce yet another relation between pairs of rings and ideals.

Definition 1.8. Let (R, I) and (S, J) be pairs as in Definition 1.7. We say that (R, I) is a *generalized localization* of (S, J) if there is a sequence of pairs

(S_i, J_i), $1 \le i \le n$, with $(S_1, J_1) = (S, J)$, $(S_n, J_n) = (R, I)$, such that for each $1 \le i \le n-1$, one of the following conditions is satisfied:

(i) $(S_{i+1}, J_{i+1}) = ((S_i)_p, (J_i)_p)$ for some $p \in \operatorname{Spec}(S_i)$ such that S_i/p is regular.
(ii) (S_{i+1}, J_{i+1}) is a deformation of (S_i, J_i).
(iii) (S_{i+1}, J_{i+1}) is a specialization of (S_i, J_i).
(iv) $(S_i, J_i) \cong (S_{i+1}(Z), J_{i+1}S(Z))$ for some finite set of variables Z over S_{i+1}.

Next we have to record in which way linkage is compatible with the operations in Definitions 1.7 and 1.8.

Lemma 1.9 [17, 2.12]. *Let S be a local Gorenstein ring, let J be a Cohen-Macaulay S-ideal of* grade g, *and let $\boldsymbol{\beta} = \beta_1, \dots, \beta_g$ be an S-regular sequence contained in J with $(\boldsymbol{\beta}) \ne J$. Let (R, I) be a specialization of (S, J) with respect to the regular sequence $\mathbf{a}$, and let $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g \subset I$ be the image of $\boldsymbol{\beta}$ under the specialization modulo $(\mathbf{a})$.*

If $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ form an R-regular sequence, then $(R, (\boldsymbol{\alpha}) : I)$ is a specialization of $(S, (\boldsymbol{\beta}) : J)$ with respect to $\mathbf{a}$.

Lemma 1.10. *In addition to the assumptions of Lemma 1.9, suppose that $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ form an R-regular sequence and that the link $I \sim (\boldsymbol{\alpha}) : I$ is geometric.*

Then also the link $J \sim (\boldsymbol{\beta}) : J$ is geometric, and $(R, I + ((\boldsymbol{\alpha}) : I))$ is a specialization of $(S, J + ((\boldsymbol{\beta}) : J))$ with respect to $\mathbf{a}$.

Proof. By Lemma 1.9,

$$(R, I + ((\boldsymbol{\alpha}) : I)) \cong (S/(\mathbf{a}), (J, (\boldsymbol{\beta}) : J, \mathbf{a})/(\mathbf{a})).$$

In particular,

$$\operatorname{grade}(J + ((\boldsymbol{\beta}) : J)) \ge \operatorname{grade}(I + ((\boldsymbol{\alpha}) : I)) = g + 1.$$

Therefore the link $J \sim (\boldsymbol{\beta}) : J$ is geometric. But then $\operatorname{grade}(J + ((\boldsymbol{\beta}) : J)) = g + 1$ (by Proposition 1.3), hence

$$\operatorname{grade}(J + ((\boldsymbol{\beta}) : J)) = \operatorname{grade}(I + ((\boldsymbol{\alpha}) : I)).$$

This together with the fact that $S/(J + ((\boldsymbol{\beta}) : J))$ is Cohen-Macaulay (by Proposition 1.3), now implies that $\mathbf{a}$ is regular on $S/(J + ((\boldsymbol{\beta}) : J))$. Therefore $(R, I + ((\boldsymbol{\alpha}) : I))$ is a specialization of $(S, J + ((\boldsymbol{\beta}) : J))$ with respect to $\mathbf{a}$. □

Lemma 1.11. *Let S be a local Gorenstein ring, let J be a licci S-ideal, let (R, I) be a generalized localization of (S, J), and assume that the residue class field of R is infinite.*

Then I is a licci R-ideal.

Proof. We use the notation of Definition 1.8. Let X be an indeterminate, then for $1 \le i \le n$ we replace (S_i, J_i) by $(S_i(X), J_i(X))$ and set $(S_{n+1}, J_{n+1}) =$

(R, I), to assume that the residue class fields of S_i are infinite for $1 \leq i \leq n+1$. Moreover, S_i are Gorenstein rings, J_i are Cohen-Macaulay S_i-ideals, and we may assume that $\operatorname{grade} J_i > 0$. Under these assumptions we have to show that the property of being licci is preserved as we pass from (S_i, J_i) to (S_{i+1}, J_{i+1}), where (S_{i+1}, J_{i+1}) is obtained from (S_i, J_i) by one of the four operations in Definition 1.8. For operation (i) the claim is trivial, for (ii) it follows from [18, 2.16], for (iii) from [33, 1.6], and for (iv) from [19, 2.12] □

Definition 1.12 [18]. Let R be a (not necessarily local) Gorenstein ring, let I be an unmixed ideal of $\operatorname{grade} g > 0$, let $\mathbf{f} = f_1, \dots, f_n$ be a generating sequence of I, let X be a g by n matrix of indeterminates over R, and consider the $R[X]$-regular sequence $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ with

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = X \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

(a) Then $L_1(\mathbf{f}) = (\boldsymbol{\alpha})R[X] : IR[X] \subset R[X]$ is called the *first generic link* of I (with respect to $\mathbf{f}$).
(b) Let R be local, and let I and g be either as above, or $I = R$ and $g > 0$ arbitrary. Then $L^1(\mathbf{f}) = (\boldsymbol{\alpha})R(X) : IR(X) \subset R(X)$ is called the *first universal link* of I (with respect to $\mathbf{f}$).

In [18, 2.11], we showed that up to equivalence of pairs in the sense of Definition 1.6(b), the definition of $(R[X], L_1(\mathbf{f}))$ only depends on I, but not on the choice of $\mathbf{f}$. We therefore write $L_1(I)$ instead of $L_1(\mathbf{f})$, and iterate the above process in order to define $L_i(I) = L_1(L_{i-1}(I))$ for $i \geq 1$ (set $L_0(I) = I$). We call $L_i(I)$ an ith *generic link* of I. We also proved that up to generic equivalence of pairs in the sense of Definition 1.6(c), $(R(X), L^1(\mathbf{f}))$ is uniquely determined by I. Thus we write $L^1(I) = L^1(\mathbf{f})$, and call $L^i(I) = L^1(L^{i-1}(I))$ $(i \geq 1)$ an ith *universal link* of I (set $L^0(I) = I$).

The next result describes the main property of generic linkage.

Theorem 1.13 [18, 2.17]. *Let (R, m) be a local Gorenstein ring, let I be a Cohen-Macaulay R-ideal with* $\operatorname{grade} I > 0$, *and let $I = I_0 \sim I_1 \sim \cdots \sim I_n$ be any sequence of links in $R = R_0$. For $1 \leq i \leq n$ consider ith generic links $L_i(I) \subset R_i$, where R_i is a polynomial ring over R_{i-1}. Set $S = R_n$.*

Then there exists a prime ideal q in S with $m \subset q$, such that for all $1 \leq i \leq n$, $(S_q, L_i(I)S_q)$ is deformation of (R, I_i).

For the purpose of this paper however, we need to have better control over the location of the prime ideal q in Theorem 1.13. This information is contained in the next theorem.

Theorem 1.14. *Let (R, m) be a local Gorenstein ring, and let I and J be two linked Cohen-Macaulay R-ideals of* $\operatorname{grade} g > 0$. *Let $\mathbf{f} = f_1, \dots, f_n$ be a* *generating sequence of I, let X be a generic g by n matrix, let*

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = X \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

and consider the first generic link $L_1(I) = L_1(\mathbf{f}) = (\boldsymbol{\alpha})R[X] : IR[X]$.

Then there exists a prime ideal q in $R[X]$ such that $m \subset q \subset (m, X)$, $(R[X]/q)_{(m,X)}$ is regular, and $(R[X]_q, L_1(I)_q)$ is a deformation of $(R(Z), JR(Z))$ for some finite set of variables Z. In particular, (R, J) is a generalized localization of $(R[X]_{(m,X)}, L_1(I)_{(m,X)})$.

Proof. Let $\beta_1, \dots, \beta_g$ be an R-regular sequence defining the link $I \sim J$, and let A be a g by n matrix with

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

Only by using elementary row operations and hence without changing the sequence $f_1, \dots, f_n$ and the ideal $(\beta_1, \dots, \beta_g)R$, we may assume that

$$A = \left(\begin{array}{c|c} I_{t\times t} & B \\ \hline 0 & C \end{array}\right)$$

where $I_{t\times t}$ is the t by t identity matrix, B is a t by $n-t$ matrix, and C is a $g-t$ by $n-t$ matrix with entries in m. In the same fashion, we decompose

$$X = \left(\begin{array}{c|c} Z & U \\ \hline V & W \end{array}\right)$$

where Z is a t by t matrix. Now set

$$q = (m, U - ZB, V, W)R[X].$$

Then q has the following properties: q is a prime ideal in $R[X]$, $R[X]/q \cong R/m[Z]$ is regular, and $m \subset q \subset (m, X)$. Then clearly (R, J) will be a generalized localization of $(R[X]_{(m,X)}, L_1(I)_{(m,X)})$ once we have shown that $(R[X]_q, L_1(I)_q)$ is a deformation of $(R(Z), JR(Z))$.

Since the entries of the matrix C are in m, and from the definition of q, it follows that the sequence $\mathbf{a}$ consisting of the entries of the matrices $U - ZB$, V, $W - C$ are contained in q, and hence forms an $R[X]_q$-regular sequence. Moreover, $R[X]_q/(\mathbf{a}) \cong R(Z)$. Now let $\Pi : R[X]_q \to R(Z)$ denote the reduction modulo $(\mathbf{a})$. Then

$$\Pi(X) = \left(\begin{array}{c|c} Z & ZB \\ \hline 0 & C \end{array}\right) = \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & I_{g-t\times g-t} \end{array}\right) A.$$

In particular,

$$\begin{pmatrix} \Pi(\alpha_1) \\ \vdots \\ \Pi(\alpha_g) \end{pmatrix} = (\Pi(X)) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & I_{g-t \times g-t} \end{array}\right) A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$= \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & I_{g-t \times g-t} \end{array}\right) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix}.$$

However, since the matrix Z is invertible over $R(Z)$, we conclude that

$$\Pi(\boldsymbol{\alpha}) = \Pi(\alpha_1), \dots, \Pi(\alpha_g)$$

form an $R(Z)$-regular sequence which generates the same $R(Z)$-ideal as $\beta_1, \dots, \beta_g$. Then Lemma 1.9 implies that $(R[X]_q, (\alpha_1, \dots, \alpha_g)R[X]_q : IR[X]_q)$ is a deformation of $(R(Z), (\beta_1, \dots, \beta_g)R(Z) : IR(Z))$. But the first pair is $(R[X]_q, L_1(I)_q)$, whereas the second pair is $(R(Z), JR(Z))$. □

Combining the proof of Theorem 1.14 and Lemma 1.10 we obtain immediately:

Corollary 1.15. *In addition to the assumptions of Theorem* 1.14, *suppose that the link* $I \sim J$ *is geometric.*

Then $(R[X]_q, (IR[X] + L_1(I))_q)$ *is a deformation of* $(R(Z), (I+J)R(Z))$. *In particular,* $(R, I+J)$ *is a generalized localization of*

$$(R[X]_{(m,X)}, (IR[X] + L_1(I))_{(m,X)}).$$

Remark 1.16. Under the assumptions of Corollary 1.15, consider the first universal link $L^1(I) = L^1(\mathbf{f}) = L_1(\mathbf{f})R(X)$.

Then $(R(X), IR(X) + L^1(I))$ is essentially a deformation of $(R, I+J)$.

Proof. The claim follows from Corollary 1.15, since $m \subset q$ and hence $R(X)$ is a localization of $R[X]_q$. □

We also remark that instead of considering just one link $I \sim J$, one could easily generalize Theorem 1.14, Corollary 1.15, and Remark 1.16 to statements concerning a whole sequence of links (similar to Theorem 1.13). The next observation will play a crucial role in the second section of this paper, since it enables us to modify a given link by using generalized localizations.

Corollary 1.17. *Let* (R, m) *be a local Gorenstein ring, and let* I *be a Cohen-Macaulay R-ideal. We consider links* $I \sim J$, *and* $I \sim K$, *and assume that the regular sequence defining the link* $I \sim K$ *is contained in* mI.

Then (R, J) *is a generalized localization of* (R, K). *If moreover both links* $I \sim J$ *and* $I \sim K$ *are geometric, then* $(R, I+J)$ *is a generalized localization of* $(R, I+K)$.

Proof. If $\operatorname{grade} I = 0$, then $K = J$, and the claim is obviously true. So let $\operatorname{grade} I > 0$.

We use the notation of Theorem 1.14, and set $S = R[X]_{(m,X)}$. Then by Theorem 1.14, (R, J) is a generalized localization of $(S, L_1(I)S)$, and by Corollary 1.15, $(R, I+J)$ is a generalized localization of $(S, IS + L_1(I)S)$ in case the link $I \sim J$ is geometric.

On the other hand, let $\boldsymbol{\beta} = \beta_1, \dots, \beta_g$ be the regular sequence defining the link $I \sim K$. Then there exists a g by n matrix A with entries in m such that

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

Now the entries of the matrix $X - A$ are contained in the maximal ideal of S, and hence form an S-regular sequence. Modulo this sequence, $\alpha_1, \dots, \alpha_g$ specializes to the R-regular sequence $\beta_1, \dots, \beta_g$. Therefore $(S, L_1(I)S) = (S, (\alpha_1, \dots, \alpha_g)S : IS)$ is a deformation of $(R, K) = (R, (\beta_1, \dots, \beta_g) : I)$ (by Lemma 1.9), and $(S, IS + L_1(I)S)$ is a deformation of $(R, I+K)$ in case the link $I \sim K$ is geometric (by Lemma 1.10). □

Remark 1.18. Given any link $I \sim J$ as in Corollary 1.17, one can always find a link $I \sim K$ such that the regular sequence defining the link $I \sim K$ is contained in mI and such that the link $I \sim K$ is geometric in case $I \sim J$ is geometric.

2. Sums of licci ideals

We are now able to prove one of our main results.

Theorem 2.1. *Let R be a local Gorenstein ring with infinite residue class field, and let I and J be geometrically linked licci R-ideals.*

Then $I + J$ is licci.

Proof. Our assumptions imply that $\operatorname{grade} I > 0$. Since I and J are linked and I is licci, there is a sequence of links in R, $J \sim I = I_0 \sim I_1 \sim \cdots \sim I_n$ such that I_n is a complete intersection.

We will prove the theorem by induction on n. If $n = 0$, then $I = I_n$ is a complete intersection, and as $I + J/I \cong \omega_{R/I}$ is cyclic (by Proposition 1.2(c)), it follows that $I + J$ is a complete intersection. Therefore we may assume that $n > 0$ and that our result is true for smaller n.

By m we denote the maximal ideal of R. In $R[X]$, consider a sequence of generic links of I, $IR[X] \sim L_1(I)R[X] \sim \cdots \sim L_n(I)R[X]$. Then by Theorem 1.13, there exists a prime ideal q in $R[X]$ with $m \subset q$, such that $(R[X]_q, L_n(I)R[X]_q)$ is a deformation of (R, I_n). In particular, $L_n(I)R[X]_q$ is a complete intersection. Moreover, $JR[X]_q$ and $IR[X]_q$ are linked, and $(R[X]_q, IR[X]_q + JR[X]_q)$ is a deformation of $(R, I+J)$. Hence by Lemma 1.11 it suffices to prove that $IR[X]_q + JR[X]_q$ is licci. Now we may replace $J \sim I = I_0 \sim I_1 \sim \cdots \sim I_n$ by the sequence of links

$$JR[X]_q \sim IR[X]_q = I_0R[X]_q \sim L_1(I)R[X]_q \sim \cdots \sim L_n(I)R[X]_q.$$

Since I and J are geometrically linked, I is generically a complete intersection, and hence the link $IR[X] \sim L_1(I)R[X]$ is geometric [17, 2.5]. In particular, $IR[X]_q$ and $L_1(I)R[X]_q$ are geometrically linked. Returning to our original notation, we may therefore assume that in the sequence of links $J \sim I = I_0 \sim I_1 \sim \cdots \sim I_n$, I and I_1 are geometrically linked. From the induction hypothesis, applied to I_1, we then conclude that $I + I_1$ is licci. Now our theorem follows from the next Proposition 2.2 and Lemma 1.11. □

Proposition 2.2. *Let R be a local Gorenstein ring, and let $J \sim I \sim L$ be two geometric links of Cohen-Macaulay R-ideals of positive grade.*

Then $(R, I+J)$ is a generalized localization of (R, H), where H is an ideal doubly linked to $I + L$.

Proof. Let $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ be an R-regular sequence defining the geometric link $I \sim L$. Since $\dim R \geq \operatorname{grade}(I + L) = g + 1$, one can easily see that there exist elements $x_1, \dots, x_g$ in m, the maximal ideal of R, such that $x_1\alpha_1, \dots, x_g\alpha_g$ form an R-regular sequence and moreover, $x_i \notin p$ for all $p \in \operatorname{Ass}(R/(\boldsymbol{\alpha}))$ and all $1 \leq i \leq g$. In particular, if we set $x = \prod_{i=1}^{g} x_i$, then $\alpha_1, \dots, \alpha_g, x$ from an R-regular sequence. Since the link $I \sim L$ defined by $\boldsymbol{\alpha}$ is geometric, it follows that for all $p \in \operatorname{Ass}(R/I)$, $I_p = (\alpha_1, \dots, \alpha_g)R_p$, and hence $I_p = (x_1\alpha_1, \dots, x_g\alpha_g)R_p$ because $x_i \notin p$. Thus if we set $K = (x_1\alpha_1, \dots, x_g\alpha_g) : I$, then I and K are geometrically linked with respect to $x_1\alpha_1, \dots, x_g\alpha_g$.

However, since the regular sequence $x_1\alpha_1, \dots, x_g\alpha_g$ is contained in mI, we may use Corollary 1.17 to conclude that $(R, I+J)$ is a generalized localization of $(R, I+K)$. Thus it suffices to prove that $H = I + K$ is doubly linked to $I + L$.

But now $K = (x_1\alpha_1, \dots, x_g\alpha_g) : I$ can be easily expressed in terms of $L = (\alpha_1, \dots, \alpha_g) : I$. Since

$$\begin{pmatrix} x_1\alpha_1 \\ \vdots \\ x_g\alpha_g \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix}$$

with

$$A = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_g \end{pmatrix},$$

and $\det A = \prod_{i=1}^{g} x_i = x$, it follows from [5, 6.1.11], that $K = (x_1\alpha_1, \dots, x_g\alpha_g, xL)$. In particular, $I + K = I + xL$, and it remains to show that $I + xL$ and $I + L$ are doubly linked.

To prove this claim we may pass to the factor ring $\overline{R} = R/(\alpha_1, \dots, \alpha_g)$, because both ideals in question contain the regular sequence $\alpha_1, \dots, \alpha_g$. By "$\overline{}$", we denote images in $\overline{R}$. Since I and L are geometrically linked with respect to $\alpha_1, \dots, \alpha_g$, it follows that $\overline{I} \cap \overline{L} = 0$ and $\operatorname{grade}(\overline{I} + \overline{L}) = 1$. Recall that by our choice of $x_1, \dots, x_g$, the elements $\alpha_1, \dots, \alpha_g, x$ form an R-regular sequence, and hence $\overline{x}$ is regular on $\overline{R}$. Moreover, $\overline{I} + \overline{x}\overline{L}$ and $\overline{I} + \overline{L}$ are unmixed ideals of grade one in a local Gorenstein ring. In this situation, the next Lemma 2.3 implies that $\overline{I} + \overline{x}\overline{L}$ and $\overline{I} + \overline{L}$ are doubly linked in $\overline{R}$. □

Lemma 2.3. *Let R be a Noetherian local ring, let I and L be R-ideals such that $I \cap L = 0$ and $\operatorname{grade}(I + L) \geq 1$, and let x be an R-regular element.*

Then there are elements $a \in I$ and $b \in L$ such that $a + b$ is an R-regular element, $a + xb$ is an R-regular element, and $(a + b)R : (I + L) = (a + xb)R : (I + xL)$.

Proof. Since $\operatorname{grade}(I + L) \geq 1$ and x is regular on R, it follows that $\operatorname{grade}(I + xL) \geq 1$. Thus $I + xL$ contains an R-regular element, which automatically has the form $a + xb$ with $a \in I$ and $b \in L$. Now suppose that $a + b$ is a zero divisor in R, then $a + b \in p$ for some $p \in \operatorname{Ass}(R)$. However $0 = I \cap L$, hence $\operatorname{Ass}(R) \subset \operatorname{Ass}(R/I) \cup \operatorname{Ass}(R/L)$ and we may assume that $a + b \in p$ for some $p \in \operatorname{Ass}(R/I)$. But also $a \in p$, hence $b \in p$, and therefore $a + xb \in p$, which is impossible since $a + xb$ is R-regular and $p \in \operatorname{Ass}(R)$.

Thus we have seen that $a + b$ and $a + xb$ are R-regular elements.

To prove the equality of the colon ideals, first observe that $I \cap xL \subset I \cap L = 0$. Therefore $I + L = I \oplus L$ and $I + xL = I \oplus xL$. But then $r \in (a + b)R : (I + L)$ if and only if for all $c \in I$ and $d \in L$ there exists an element $s \in R$ such that $cr = as$ and $dr = bs$. Since x is R-regular, the latter condition is equivalent to the statement that for all $c \in I$ and $xd \in xL$ there exists an element $s \in R$ such that $cr = as$ and $xdr = xbs$. This in turn holds true if and only if $r \in (a + xb)R : (I + xL)$. □

Remark 2.4. Let R be a local Gorenstein ring, and let $I_0 \sim I_1 \sim \cdots \sim I_{n-1} \sim I_n$ be a sequence of geometric links of Cohen-Macaulay R-ideals with positive grade. Then repeated application of Proposition 2.2 implies that $(R, I_0 + I_1)$ and $(R, I_{n-1} + I_n)$ share any property which is preserved under even linkage and generalized localization.

The second main result in this section will contain a partial coverse of Theorem 2.1. We will prove that every licci Gorenstein ideal (of grade at least two in a local Gorenstein ring R with infinite residue class field) is a generalized localization of a sum of two geometrically linked licci R-ideals (Theorem 2.17). Theorem 2.17 is in some sense the best possible result, since one cannot expect that every licci Gorenstein ideal is actually equal to a sum of two geometrically linked licci ideals. To see this we provide the following obstruction for an ideal to be the sum of linked ideals:

Proposition 2.5. *Let R be a regular local ring, and let I and J be two geometrically linked licci R-ideals which are not both complete intersections.*

Then $R/I + J$ does not satisfy Serre's condition (R_3). Moreover, if $R/I + J$ satisfies (R_2), then $R/I + J$ is not factorial.

Proof. We may assume that I is not a complete intersection. In the first part of the proof we reduce the problem to the case where J is a complete intersection.

Consider ith universal links of I, $L^i(I) \subset R^i$, where $L^0(I) = I \subset R = R^0$, and for $i > 0$, R^i is obtained from R^{i-1} by a purely transcendental extension of the residue class field. Let $n > 0$ such that $L^n(I) \neq R^n$, then also $L^i(I) \neq R^i$ for $0 \leq i \leq n$ (cf. [18, 2.13]). In particular, $L^{i-1}(I)R^i$ and $L^i(I)$ are linked in R^i, and these links are geometric since I is generically a complete intersection (cf. [17, 2.5]).

We will prove by induction on $n > 0$, that $(R^n, L^{n-1}(I)R^n + L^n(I))$ is essentially a deformation of $(R, I + J)$. But for $n = 1$, this is the content of Remark 1.16. So let $n > 1$. We may apply our result for $n = 1$ to the geometric link $L^{n-1}(I) \sim L^{n-2}(I)R^{n-1}$ and conclude that $(R^n, L^{n-1}(I)R^n + L^1(L^{n-1}(I)))$ $= (R^n, L^{n-1}(I)R^n + L^n(I))$ is essentially a deformation of $(R^{n-1}$, $L^{n-1}(I) + L^{n-2}(I)R^{n-1})$, which by induction hypothesis, is essentially a deformation of $(R, I + J)$.

Since I is licci, but not a complete intersection, it follows from [18, 2.17], that there exists an $n > 0$ such that $L^n(I)$ is a complete intersection, but $L^{n-1}(I)$ is not a complete intersection. Now suppose that $R/I + J$ satisfies (R_3), or that $R/I+J$ is (R_2) and factorial. Then since $(R^n, L^{n-1}(I)R^n + L^n(I))$ is essentially a deformation of $(R, I + J)$, we conclude that

$$R^n/(L^{n-1}(I)R^n + L^n(I))$$

also would have to be (R_3) or factorial respectively (the factoriality is preserved under essentially a deformation since $R/I+J$ satisfies (R_2), cf. [25, $(D-S)'$]). Thus replacing the link $I \sim J$ by the link $L^{n-1}(I)R^n \sim L^n(I)$, we may from now on assume that I is not a complete intersection, but J is a complete intersection.

Since I is directly linked to a complete intersection, but is not a complete intersection itself, there exists a prime ideal p in R with $I \subset p$ such that $\dim(R/I)_p \leq 4$, and I_p is not Gorenstein (cf. [18, 3.1 or 4.2(b)]). Then automatically $I + J \subset p$, and $\dim(R/I + J)_p \leq 3$. Thus replacing R by R_p, it suffices to show that $R/I + J$ is not factorial (and hence not regular). By the choice of p we may assume that I is not Gorenstein.

So let $x_1, \ldots, x_g$ be a regular sequence generating the complete intersection J, let $\alpha_1, \ldots, \alpha_g$ be a regular sequence defining the link $I \sim J$, and write

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_g \end{pmatrix}$$

where A is a g by g matrix with entires in R. We may even assume that

$$A = \left(\begin{array}{c|c} I_{g-t \times g-t} & 0 \\ \hline 0 & B \end{array}\right)$$

with B a t by t matrix whose entries are contained in m, the maximal ideal of R. Since $I = (\alpha_1, \ldots, \alpha_g) : (x_1, \ldots, x_g)$ is not Gorenstein, it follows that $2 \leq r(R/I) = \nu((x_1, \ldots, x_g)/(\alpha_1, \ldots, \alpha_g)) = t$ (cf. Proposition 1.2(c)). On the other hand, $I = (\alpha_1, \ldots, \alpha_g, \det A) = (\alpha_1, \ldots, \alpha_g, \det B)$ (e.g. [18, 3.1]), and therefore $I + J = (x_1, \ldots, x_g, \det B)$. Now $R/I + J$ cannot be factorial, because $\det B$ is the determinant of a matrix whose entries are in m and whose size is at least 2 (cf. [1] or [7, the proof on p. 124]). □

Remark 2.6. In contrast to Proposition 2.5, it is easy to construct rings of the form $R/I + J$ that do satisfy (R_2). Let R' be a regular local ring, let I' be a licci R'-ideal of positive grade such that R'/I' satisifes (R_3) (at least after deformation, the latter condition holds for any licci ideal, cf. [32]), let $L_1(I') \subset R'[X]$ be a first generic link of I', $q \in \operatorname{Spec}(R'[X])$ with $I'R'[X] + L_1(I') \subset q$, $R = R'[X]_q$, $I = I'R$, and $J = L_1(I')R$. Then I and J are geometrically linked licci R-ideals and $R/I+J$ satisfies (R_2) (the latter claim was essentially proved in [17, 2.8 and 2.9]).

Corollary 2.7. *Let R be a regular local ring with infinite residue class field, and let K be an R-ideal which is not a complete intersection.*

Then (R, K) has a deformation $(\widetilde{R}, \widetilde{K})$ such that $\widetilde{K}$ is not the sum of two geometrically linked licci $\widetilde{R}$-ideals.

Proof. We may assume that K is the sum of two geometrically linked licci R-ideals. But then K itself is licci by Theorem 2.1, and hence (R, K) has a deformation $(\widetilde{R}, \widetilde{K})$ such that $\widetilde{R}/\widetilde{K}$ satisfies (R_3) (cf. [32]). Now suppose that $\widetilde{K}$ is the sum of two geometrically linked licci $\widetilde{R}$-ideals I and J. Neither I nor J could be a complete intersection since $\widetilde{K}$ is not a complete intersection. But then Proposition 2.5 implies that $\widetilde{R}/\widetilde{K}$ could not be (R_3). □

Example 2.8. Let R be a regular local ring, let K be an R-ideal of grade 3 which is not a complete intersection, and assume that R/K satisfies (R_3).

Then K is not the sum of two geometrically linked Cohen-Macaulay R-ideals.

Proof. The claim follows immediately from Proposition 2.5, since the Cohen-Macaulay ideals of grade 2 in a regular local ring are automatically licci, [2], [8]. □

We now aim at proving our second main result, which says that every licci Gorenstein ideal (of grade at least two in a local Gorenstein ring with infinite residue class field) is a generalized localization of a sum of two geometrically linked licci ideals (Theorem 2.17). In the proof of the theorem, we will proceed by induction on the number of steps necessary to link the licci Gorenstein ideal to a complete intersection. To do so, we first have to prove that we can "lift" links through generalized localizations and still retain certain properties, such as Gorensteinness. This is the content of a set of lemmas which follow now and in particular of Lemma 2.14 and Corollary 2.15 (it is also for the proof of Lemma

2.14 that we need the regularity assumption in Definition 1.8(i)). A second ingredient in the proof of Theorem 2.17 is Proposition 2.16 which establishes the ubiquity of tight double links (cf. Definition 1.5) and which enables us to replace double links of Gorenstein ideals by tight double links.

Lemma 2.9. *Let (R, m) be a local Cohen-Macaulay ring, and let $I = (f_1, \ldots, f_n)$ be an R-ideal with* $\operatorname{grade} I \geq g > 0$.

Then there are elements $a_{ij} \in m$, $1 \leq i \leq g$, $1 \leq j \leq n$, such that for $\gamma_i = f_i + \sum_{j=1}^n a_{ij} f_j$, $\gamma_1, \ldots, \gamma_g$ form an R-regular sequence.

Proof. It suffices to treat the case $g = 1$. Write $\operatorname{Ass}(R) = \{q_1, \ldots, q_r, q_{r+1}, \ldots, q_s\}$ where $0 \leq r \leq s$, such that $f_1 \in q_i$ for $1 \leq i \leq r$, and $f_1 \notin q_i$ for $r+1 \leq i \leq s$. Then $mI \cap \bigcap_{i=r+1}^s q_i \not\subset \bigcup_{i=1}^r q_i$, and we take $\gamma_1 = f_1 + h$ with $h \in mI$, $h \in q_i$ for $r+1 \leq i \leq s$, and $h \notin q_i$ for $1 \leq i \leq r$. □

Lemma 2.10. *Let R be a local Cohen-Macaulay ring, let I be an R-ideal with* $\operatorname{grade} I \geq g > 0$, *let p be a prime ideal in R with $I \subset p$, and let $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_g$ be an R_p-regular sequence contained in I_p.*

Then there exists an R-regular sequence $\boldsymbol{\beta} = \beta_1, \ldots, \beta_g$ contained in I such that $(\boldsymbol{\beta})R_p = (\boldsymbol{\alpha})R_p$.

Proof. We may assume that $g = 1$. Obviously

$$I \cap (\alpha R_p) \not\subset (R \cap (p\alpha R_p)) \cup \bigcup_{q \in \operatorname{Ass}(R)} q,$$

and hence there exists an R-regular element β in I with $\beta/1 \in \alpha R_p \backslash p\alpha R_p$. □

Lemma 2.11. *Let R be a local Gorenstein ring, let $I \sim J$ be a link of Cohen-Macaulay R-ideals, and let $(\overline{R}, \overline{I})$ be a specialization of (R, I).*

Then there exists a link $\overline{I} \sim \overline{J}$ of $\overline{R}$-ideals such that $(\overline{R}, \overline{J})$ and (R, J) have a common deformation.

Proof. Let m and $\overline{m}$ be the maximal ideals of R and $\overline{R}$, and set $g = \operatorname{grade} I$. If $g = 0$, then the claim follows immediately from Lemma 1.9. So assume that $g > 0$, and let $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_g$ be a regular sequence defining the link $I \sim J$. Complete $\boldsymbol{\alpha}$ to a generating sequence $\mathbf{f} = f_1, \ldots, f_n = \alpha_1, \ldots, \alpha_g, f_{g+1}, \ldots, f_n$ of I, let X be a generic g by n matrix, define a g by n matrix

$$E = (I_{g \times g} \mid 0),$$

and consider the $R[X]$-regular sequence $\boldsymbol{\beta} = \beta_1, \ldots, \beta_g$, where

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix} = (E + X) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

The entries of the matrix X form a regular sequence on $R[X]_{(m,X)}$, and modulo this sequence, $\beta_1, \ldots, \beta_g$ specialize to the R-regular sequence $\alpha_1, \ldots, \alpha_g$. Therefore it follows from Lemma 1.9 that

$$(R[X]_{(m,X)}, ((\boldsymbol{\beta})R[X] : IR[X])_{(m,X)})$$

is a deformation of $(R, (\boldsymbol{\alpha}) : I) = (R, J)$.

By " $\overline{}$ " we denote images of elements of R in the factor ring $\overline{R}$. Then $\overline{f}_1, \ldots, \overline{f}_n$ generate the ideal $\overline{I}$, and since $E+X$ is still a generic matrix over $\overline{R}$, it follows that

$$\begin{pmatrix} \overline{\beta}_1 \\ \vdots \\ \overline{\beta}_g \end{pmatrix} = (E+X) \begin{pmatrix} \overline{f}_1 \\ \vdots \\ \overline{f}_n \end{pmatrix}$$

form a regular sequence on $\overline{R}[X]_{(\overline{m},X)}$. But then again Lemma 1.9 implies that

$$(\overline{R}[X]_{(\overline{m},X)}, ((\overline{\boldsymbol{\beta}})\overline{R}[X] : \overline{I}\,\overline{R}[X])_{(\overline{m},X)})$$

is a specialization of

$$(R[X]_{(m,X)}, ((\boldsymbol{\beta})R[X] : IR[X])_{(m,X)}).$$

By Lemma 2.9 there exists a g by n matrix F with entries in $\overline{m}$ such that for

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_g \end{pmatrix} = (E+F) \begin{pmatrix} \overline{f}_1 \\ \vdots \\ \overline{f}_n \end{pmatrix},$$

$\gamma = \gamma_1, \ldots, \gamma_g$ form an $\overline{R}$-regular sequence. On the other hand, the entries of the matrix $X-F$ are contained in the maximal ideal of $\overline{R}[X]_{(\overline{m},X)}$, and hence form a regular sequence. Modulo this sequence, $\overline{\beta}_1, \ldots, \overline{\beta}_g$ specialize to the $\overline{R}$-regular sequence $\gamma_1, \ldots, \gamma_g$, and thus again by Lemma 1.9, $(\overline{R}, (\gamma) : \overline{I})$ is a specialization of

$$(\overline{R}[X]_{(\overline{m},X)}, ((\overline{\boldsymbol{\beta}})\overline{R}[X] : \overline{I}\,\overline{R}[X])_{(\overline{m},X)}).$$

Now define $\overline{J} = (\gamma) : \overline{I}$. We have seen above that $(\overline{R}, \overline{J})$ and (R, J) have

$$(R[X]_{(m,X)}, ((\boldsymbol{\beta})R[X] : IR[X])_{(m,X)})$$

as a common deformation. □

Corollary 2.12. *Let $\widetilde{R}$ be a local Gorenstein ring, let $\widetilde{I}$ be a Cohen-Macaulay $\widetilde{R}$-ideal, let (R, I) be a generalized localization of $(\widetilde{R}, \widetilde{I})$, and consider a link in R, $I \sim J$.*

Then there exists a link in $\widetilde{R}$, $\widetilde{I} \sim \widetilde{J}$, such that (R, J) is a generalized localization of $(\widetilde{R}, \widetilde{J})$.

Proof. We may assume that (R, I) is obtained from $(\widetilde{R}, \widetilde{I})$ by one of the four operations in Definition 1.8, and we have to show that in each of the four cases, the link $I \sim J$ can be "lifted" to a link $\widetilde{I} \sim \widetilde{J}$.

But for operation (i) (I is a localization of $\widetilde{I}$), this follows from Lemma 2.10, for (ii) ($\widetilde{I}$ is a specialization of I), from Lemma 2.11, for (iii) ($\widetilde{I}$ is a deformation of I), from [18, 2.16], and for (iv) ($\widetilde{I} = IR(Z)$), the claim is trivial. □

Lemma 2.13. *Let R be a regular local ring with infinite residue class field and* $\dim R \geq 1$, *and let $I \neq 0$ be an R-ideal.*

Then there exists a prime ideal q of R such that R/q is a discrete valuation ring, and $I \not\subset q$.

Proof. Let m be the maximal ideal of R. By induction on $\dim R$ it suffices to prove the following statement: If $\dim R \geq 2$, then there exists an element $x \in m\backslash m^2$ such that $I \not\subset (x)$. To this end we only have to show that there are infinitely many elements in $m\backslash m^2$ which are not associates. So let z_1, z_2 be part of a regular system of parameters of R, and for every element $a \in R/m$ choose a preimage b in R and set $x_a = z_1 + bz_2$. Now $\{x_a | a \in R/m\}$ is the desired set of elements. □

Lemma 2.14. *Let R be a local Gorenstein ring with infinite residue class field, let I be a Cohen-Macaulay R-ideal of* grade g, *let p be a prime ideal in R with $I \subset p$ and R/p regular, consider a link in R_p, $I_p \sim J$, further assume that $\nu(I) = g + 1 = \nu(I_p)$ and that J is Gorenstein.*

Then there exists a link in R, $I \sim K$ such that K is Gorenstein, and (R_p, J) is a generalized localization of (R, K).

Proof. If $g = 0$, simply take $K = 0 : I$. Thus we may assume that $g > 0$. We may also suppose that $p \neq m$, where m denotes the maximal ideal of R.

By Lemma 2.10 there is an R-regular sequence $\boldsymbol{\beta} = \beta_1, \ldots, \beta_g$ in I such that the R_p-regular sequence $\boldsymbol{\beta}/1 = \beta_1/1, \ldots, \beta_g/1$ defines the link $I_p \sim J$. If $f = f_1, \ldots, f_{g+1}$ is a minimal generating sequence of I, then there exists a g by $g+1$ matrix A with entries in R such that

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_{g+1} \end{pmatrix}.$$

Note that $\mathbf{f}/1 = f_1/1, \ldots, f_{g+1}/1$ is also a minimal generating sequence of I_p and that

$$\begin{pmatrix} \frac{\beta_1}{1} \\ \vdots \\ \frac{\beta_g}{1} \end{pmatrix} = A \begin{pmatrix} \frac{f_1}{1} \\ \vdots \\ \frac{f_{g+1}}{1} \end{pmatrix}.$$

Since $J = (\beta_1/1, \ldots, \beta_g/1)R_p : I_p$ is Gorenstein by assumption, it follows that $\omega_{R_p/J} \cong I_p/(\beta_1/1, \ldots, \beta_g/1)R_p$ is cyclic (Proposition 1.2(c)). But as $\frac{f_1}{1}, \ldots, \frac{f_{g+1}}{1}$ minimally generate I_p, we then conclude that one of the maximal minors of A has to be invertible in R_p. In other words, we have seen that $I_g(A) \not\subset p$, where $I_g(A)$ denotes the R-ideal generated by the g by g minors of A.

Since R/p is a regular local ring with infinite residue class field and $\dim R/p \geq 1$ and since $I_g(A) \not\subset p$, it follows from Lemma 2.13 that there exists a prime ideal q of R, $p \subset q$, such that R/q is a discrete valuation ring and $I_g(A) \not\subset q$. But then one of the maximal minors of A is invertible in R_q, hence $I_q/(\beta_1/1, \ldots, \beta_g/1)R_q$ is cyclic, and therefore $(\beta_1/1, \ldots, \beta_g/1)R_q : I_q$ is Gorenstein. Thus replacing the link $I_p \sim (\beta_1/1, \ldots, \beta_g/1)R_p : I_p$ by the link $I_q \sim (\beta_1/1, \ldots, \beta_g/1)R_q : I_q$, we may from now on assume that R/p is a discrete valuation ring.

Let Π be an element in R whose image in R/p is a uniformizing parameter of R/p. Then the matrix A can be written as $A = B + C$ where B and C are g by $g+1$ matrices, the entries of C are in p, and the entries of B are of the form $u_{ij}\Pi^{n_{ij}}$ with $0 \leq n_{ij} \leq \infty$ and u_{ij} units in R. Note that $I_g(B) \not\subset p$ since $I_g(A) \not\subset p$.

Because of the special form of B we may multiply A from the right by a $g+1$ by $g+1$ matrix which is invertible over R, to assume that

$$B = \left(D \mid 0\right),$$

where D is a g by g matrix with $\det D \notin p$. (Doing so we also obtain a new generating set **f** of I, which we still call **f**.) Now consider the g by $g+1$ matrix $E = (I_{g\times g}|0)$. Then $B = DE$, hence $A = DE + C$.

On the other hand, it follows from Lemma 2.9 that there exists a g by $g+1$ matrix F with entries in m such that

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_g \end{pmatrix} = (E+F) \begin{pmatrix} f_1 \\ \vdots \\ f_{g+1} \end{pmatrix}$$

is an R-regular sequence. We now define $K = (\gamma_1, \ldots, \gamma_g)R : I$. Then I and K are linked with respect to $\gamma_1, \ldots, \gamma_g$, and K is Gorenstein since $I_g(E+F) = R$.

It remains to show that (R_p, J) is a generalized localization of (R, K). To this end consider a generic g by $g+1$ matrix X, the $R[X]$-regular sequence

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = (E+X) \begin{pmatrix} f_1 \\ \vdots \\ f_{g+1} \end{pmatrix},$$

and the ideal $H = (\alpha_1, \ldots, \alpha_g)R[X]_{(m,X)} : IR[X]_{(m,X)}$. Then $IR[X]_{(m,X)}$ and H are linked with respect to $\alpha_1, \ldots, \alpha_g$.

It suffices to prove that $(R[X]_{(m,X)}, H)$ is a deformation of (R, K), and that (R, J) is a specialization of $(R[X]_{(p,X)}, H_{(p,X)})$. To show both claims we use Lemma 1.9. For the first claim simply notice that the entries of the matrix F are in m, hence the entries of $X - F$ are contained in the maximal ideal of $R[X]_{(m,X)}$ and form a regular sequence, and that modulo this sequence, $\alpha_1, \ldots, \alpha_g$ specialize to the R-regular sequence $\gamma_1, \ldots, \gamma_g$. To prove the second claim we remark that the matrix D is invertible over $R[X]_{(p,X)}$ (since $\det D \notin p$), and the entries of C are contained in p, hence the entries of $X - D^{-1}C$ are contained in the maximal ideal of $R[X]_{(p,X)}$ and form a regular sequence. Modulo this sequence, $\alpha_1, \ldots, \alpha_g$ specialize to

$$(E + D^{-1}C)\begin{pmatrix} f_1 \\ \vdots \\ f_{g+1} \end{pmatrix} = D^{-1}(DE + C)\begin{pmatrix} f_1 \\ \vdots \\ f_{g+1} \end{pmatrix}$$
$$= D^{-1}A\begin{pmatrix} f_1 \\ \vdots \\ f_{g+1} \end{pmatrix} = D^{-1}\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix}.$$

In R_p however, the regular sequence

$$D^{-1}\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix}$$

generates the same ideal as the regular sequence $\beta_1, \ldots, \beta_g$ which defines the link $I_p \sim J$. □

Corollary 2.15. *Let (S, m) be a local Gorenstein ring with infinite residue class field, let J be a Gorenstein S-ideal, let (R, I) be a generalized localization of (S, J), and consider a sequence of links in R, $I = I_0 \sim I_1 \sim I_2$ such that I_2 is a Gorenstein ideal, but I_0 and I_2 are not both complete intersections.*

Then there exists a sequence of links in S, $J = J_0 \sim J_1 \sim J_2$, such that the regular sequence defining the link $J_0 \sim J_1$ is contained in mJ_0, J_2 is a Gorenstein ideal, and (R, I_2) is a generalized localization of (S, J_2).

Proof. By Corollary 2.12, we can find a link in S, $J = J_0 \sim J_1'$, such that (R, I_1) is a generalized localization of (S, J_1') and by Remark 1.18, there exists a link $J = J_0 \sim J_1$, such that the sequence defining this link is contained in mJ_0. But then Corollary 1.17 implies that (S, J_1') is a generalized localization of (S, J_1). In particular, (R, I_1) is a generalized localization of (S, J_1).

The ideal I_1 is not a complete intersection, since otherwise the Gorenstein ideals I_0 and I_2 would have to be almost complete intersections (cf. Proposition 1.2(c)) of finite projective dimension (cf. [28]), and hence complete intersections (cf. [22]). But this is ruled out by our assumptions. So let $g = \operatorname{grade} I_1$, then $\nu(I_1) \geq g + 1$. On the other hand, J_0 is Gorenstein, hence $\nu(J_1) \leq g + 1$, and (R, I_1) is a generalized localization of (S, J_1), thus $\nu(I_1) \leq \nu(J_1)$. Therefore $\nu(J_1) = g + 1 = \nu(I_1)$. Moreover, I_2 is Gorenstein by assumption.

Now we can use Lemma 2.14, Lemma 2.11, and [18, 2.16], to conclude that there exists a link in S, $J_1 \sim J_2$ such that J_2 is Gorenstein, and (R, I_2) is a generalized localization of (S, J_2) (note that we may assume that operation (iv) in Definition 1.8 occurs at most for $i = n - 1$, hence leaving the residue class fields of S_i infinite for all $1 \leq i \leq n - 1$). □

Proposition 2.16. *Let R be a local Gorenstein ring, let $I_0 \sim I_1 \sim I_2$ be a sequence of links in R, and assume that I_0 and I_2 are Gorenstein ideals.*

Then either (R, I_0) *and* (R, I_2) *have a common deformation, or else* $I_0 \sim I_1 \sim I_2$ *is a tight double link.*

Proof. First note that if I_1 is a complete intersection, then I_0 and I_2 are also complete intersections by [22], in which case (R, I_0) and (R, I_2) have a common deformation. Therefore we may from now on assume that I_1 is not a complete intersection. We may also suppose that $g = \operatorname{grade} I > 0$ because otherwise, $(R, I_0) = (R, I_2)$.

Now let $\mathbf{a} = a_1, \dots, a_g$ be an R-regular sequence defining the link $I_0 \sim I_1$. Since I_0 is Gorenstein, it follows that $I_1/(\mathbf{a})$ is cyclic (Proposition 1.2(c)), and hence there exists an element $b \in R$ with $I_1 = (\mathbf{a}, b)$. Moreover, the elements $\mathbf{a}, b$ generate I_1 minimally because I_1 is not a complete intersection. On the other hand let $\mathbf{c} = c_1, \dots, c_g$ be an R-regular sequence defining the link $I_1 \sim I_2$. Then

$$\begin{pmatrix} c_1 \\ \vdots \\ c_g \end{pmatrix} = A \begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b \end{pmatrix}$$

for some g by $g+1$ matrix A with entries in R. Since I_2 is Gorenstein, and the elements $\mathbf{a}, b$ generate I_1 minimally, it follows as in the proof of Lemma 2.14 that after reduction modulo the maximal ideal m of R, the matrix A has rank g over R/m. Thus after elementary row operations and elementary column operations only involving the first g columns, and hence without changing the ideals $(\mathbf{a})$ and $(\mathbf{c})$ we may assume that

$$A = \left(\begin{array}{cccc|c} 1 & & & 0 & e_1 \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & \\ 0 & \dots & 0 & d & e_g \end{array}\right).$$

We now consider several cases:

Case 1. $e_g \notin m$. Then by elementary row operations we obtain

$$A = \left(\begin{array}{cccc|c} 1 & & & d_1 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & d_{g-1} & 0 \\ 0 & \dots & 0 & d & 1 \end{array}\right),$$

and after column operations only involving the first g columns we may assume that

$$A = \left(\begin{array}{cccc|c} 1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & 0 \\ 0 & \dots & 0 & d & 1 \end{array}\right).$$

Now set $\alpha = a_1, \dots, a_{g-1} = c_1, \dots, c_{g-1}$, $\beta = a_g$, and $\delta = c_g = da_g + b$. Then $I_1 = (\mathbf{a}, b) = (\alpha, \beta, \delta) = (\alpha, \beta) : I_0 = (\alpha, \delta) : I_2$, and $I_0 \sim I_1 \sim I_2$ is a tight double link.

Case 2. $e_i \notin m$ for some $1 \le i \le g-1$. We may assume that $i = 1$. Then we get

$$A = \left(\begin{array}{ccccc|c} 1 & & & & 0 & 1 \\ f_2 & 1 & & & & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ f_{g-1} & & & 1 & 0 & 0 \\ 0 & & \dots & 0 & d & e_g \end{array}\right),$$

and even

$$A = \left(\begin{array}{ccccc|c} 1 & & & & 0 & 1 \\ 0 & 1 & & & & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & & & 1 & 0 & 0 \\ 0 & & \dots & 0 & d & e_g \end{array}\right).$$

If $e_g \notin m$, then the claim follows from the first case, and if $e_g \in m$, then $d \notin m$ since one of the maximal minors of A has to be invertible. By elementary row operations we then obtain

$$A = \left(\begin{array}{ccccc|c} 1 & & & & 0 & 1 \\ 0 & 1 & & & & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & & & 1 & 0 & 0 \\ f & & \dots & 0 & 1 & 0 \end{array}\right),$$

and after changing the first g columns we may assume $f = 0$. Now set $\alpha = a_2, \dots, a_g = c_2, \dots, c_g$, $\beta = a_1$, and $\delta = c_1 = a_1 + b$. Then $I_1 = (\mathbf{a}, b) = (\alpha, \beta, \delta) = (\alpha, \beta) : I_0 = (\alpha, \delta) : I_2$.

Case 3. $e_i \in m$ for all $1 \le i \le g$. Then $d \notin m$, and we may assume that $d = 1$. Let $X = x_1, \dots, x_g$ be a set of variables, and in $S = R[X]_{(m,X)}$ consider the regular sequence $\gamma = \gamma_1, \dots, \gamma_g$ where

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_g \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & & & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_g \end{array}\right) \begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b \end{pmatrix}.$$

We define $J = (\gamma)S : I_1 S$, and claim that (S, J) is a common deformation of (R, I_0) and (R, I_2). But this is obvious from Lemma 1.9: Modulo the S-regular sequence $x_1, \dots, x_g$, the sequence $\gamma_1, \dots, \gamma_g$ specializes to the R-regular sequence $a_1, \dots, a_g$, and hence $(S, (\gamma)S : I_1 S) = (S, J)$ is a deformation of $(R, (\mathbf{a}) : I_1) = (R, I_0)$, whereas modulo the S-regular sequence

$x_1 - e_1, \ldots, x_g - e_g$ (recall that $e_i \in m$), $\gamma_1, \ldots, \gamma_g$ specializes to the R-regular sequence $c_1, \ldots, c_g$, and hence $(S, (\gamma)S : I_1S) = (S, J)$ is a deformation of

$$(R,\ (\mathbf{c}) : I_1) = (R, I_2). \quad \square$$

We are now ready to prove our second main result.

Theorem 2.17. *Let R be a local Gorenstein ring with infinite residue class field, and let K be a licci Gorenstein R-ideal with* $\operatorname{grade} K \geq 2$.

Then there exist geometrically linked licci R-ideals I and J such that (R, K) is a generalized localization of $(R, I+J)$.

Proof. Since K is a licci Gorenstein R-ideal and the residue class field of R is infinite it follows from [19, 2.6], that there exists a sequence of links in R, $K = K_0 \sim K_1 \sim K_2 \sim \cdots \sim K_{2n}$, such that K_{2n} is a complete intersection, and K_{2i} are Gorenstein ideals for $0 \leq i \leq n$. We will prove the theorem by induction on n. If $n = 0$, then $K = K_{2n}$ is a complete intersection of grade at least two, and the claim is obviously true. Thus we may assume that $n > 0$, and that K is not a complete intersection.

By induction hypothesis applied to the ideal K_2, there exist geometrically linked licci R-ideals I and J such that (R, K_2) is a generalized localization of $(R, I+J)$.

Now let m be the maximal ideal of R. By Remark 1.18, we can find a geometric link in R, $I \sim J'$, such that the R-regular sequence defining this link is contained in mI, and by Corollary 1.17, $(R, I+J)$ is a generalized localization of $(R, I+J')$. In particular, (R, K_2) is a generalized localization of $(R, I+J')$. Therefore, replacing $I \sim J$ by $I \sim J'$, we may from now on assume that the R-regular sequence defining the link $I \sim J$ is contained in mI. For the remainder of the proof it is also more convenient to change our notation, and to set $H_0 = K_2$, $H_1 = K_1$, $H_2 = K_0 = K$.

Now $\widetilde{H}_0 = I + J$ is a Gorenstein ideal in a Gorenstein ring R with infinite residue class field, (R, H_0) is a generalized localization of $(R, \widetilde{H}_0)$, and $H_0 \sim H_1 \sim H_2$ is a sequence of links in R such that H_2 is a Gorenstein ideal but not a complete intersection. Then Corollary 2.15 implies that there exists a sequence of links in R, $\widetilde{H}_0 \sim \widetilde{H}_1 \sim \widetilde{H}_2$, such that the regular sequence defining the link $\widetilde{H}_0 \sim \widetilde{H}_1$ is contained in $m\widetilde{H}_0$, $\widetilde{H}_2$ is a Gorenstein ideal, and $(R, H_2) = (R, K)$ is a generalized localization of $(R, \widetilde{H}_2)$. Thus it suffices to prove our claim for $\widetilde{H}_2$ instead of K. Replacing $H_0 \sim H_1 \sim H_2$ by $\widetilde{H}_0 \sim \widetilde{H}_1 \sim \widetilde{H}_2$ we may from now on make the following assumptions:

We are given a sequence of links in R, $H_0 \sim H_1 \sim H_2$, where H_0 and H_2 are Gorenstein ideals of $\operatorname{grade} g+1$, and moreover,

(2.18) $H_0 = I + J$, where I and J are geometrically linked licci R-ideals of $\operatorname{grade} g$,

(2.19) the regular sequence $\alpha_1, \ldots, \alpha_g$ defining the link $I \sim J$ is contained in mI, (2.20) the regular sequence defining the link $H_0 \sim H_1$ is contained in mH_0.

We will have to show that under these assumptions, (R, H_2) is a generalized localization of a sum of two geometrically linked licci R-ideals. Since this is obviously true if (R, H_0) and (R, H_2) have a common deformation, we may assume that (R, H_0) and (R, H_2) have no common deformation. But then Proposition 2.16 implies that

(2.21) $H_0 \sim H_1 \sim H_2$ is a tight double link, i.e., $H_1 = (\gamma_1, \dots, \gamma_g, \beta, \delta)$, where $\gamma_1, \dots, \gamma_g, \beta$ is an R-regular sequence defining the link $H_0 \sim H_1$ (and hence $\gamma_1, \dots, \gamma_g, \beta \subset mH_0 = mI + mJ$), and $\gamma_1, \dots, \gamma_g, \delta$ is an R-regular sequence defining the link $H_1 \sim H_2$.

We are now going to reduce the problem to the case where the regular sequences $\alpha_1, \dots, \alpha_g$ from (2.19) and $\gamma_1, \dots, \gamma_g$ from (2.21) coincide. For this part of the proof we need that $\alpha_1, \dots, \alpha_g$ and $\gamma_1, \dots; \gamma_g, \beta$ are contained in mI and mH_0 respectively.

Choose generating sequences $\mathbf{f} = f_1, \dots, f_r$ of I and $\mathbf{h} = h_1, \dots, h_s$ of J. Then $\mathbf{f}, \mathbf{h}$ form a generating sequence of $H_0 = I + J$, and because of (2.21) there exists a $g+1$ by $r+s$ matrix C with entries in m such that

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_g \\ \beta \end{pmatrix} = C \begin{pmatrix} f_1 \\ \vdots \\ f_r \\ h_1 \\ \vdots \\ h_s \end{pmatrix}.$$

Now let X be a generic $g+1$ by $r+s$ matrix, and set

$$\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{g+1} \end{pmatrix} = X \begin{pmatrix} f_1 \\ \vdots \\ f_r \\ h_1 \\ \vdots \\ h_s \end{pmatrix}.$$

In $S = R[X]_{(m,X)}$ we consider the generic link $L_1 = L_1(\mathbf{f}, \mathbf{h}) = L_1(H_0) = (\varepsilon_1, \dots, \varepsilon_{g+1})S : H_0 S$.

Note that the entries of the matrix $X - C$ are contained in the maximal ideal of S and hence form an S-regular sequence. Modulo this sequence, the S-regular sequence $\varepsilon_1, \dots, \varepsilon_{g+1}$ specializes to the R-regular sequence $\gamma_1, \dots, \gamma_g, \beta$, and therefore by Lemma 1.9, $(S, L_1) = (S, (\varepsilon_1, \dots, \varepsilon_{g+1})S : H_0 S)$ is a deformation of $(R, H_1) = (R, (\gamma_1, \dots, \gamma_g, \beta) : H_0)$. In particular, since

$H_1 = (\gamma_1, \dots, \gamma_g, \beta, \delta)$, we can find a preimage μ of δ in S such that $L_1 = (\varepsilon_1, \dots, \varepsilon_{g+1}, \mu)$.

On the other hand, $\operatorname{grade}(H_0/(\alpha_1, \dots, \alpha_g)) = 1$, and hence by Lemma 2.9, there exists an element $b \in mH_0$ such that $\alpha_1, \dots, \alpha_g, b$ form an R-regular sequence. Thus we may consider the link $H_1' = (\alpha_1, \dots, \alpha_g, b) : H_0$. Write $b = \sum_{i=1}^r a_i f_i + \sum_{i=1}^s a_{i+r} h_i$ with $a_j \in m$ for all $1 \leq j \leq r+s$, and set

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

Here A is a g by r matrix, and in the light of (2.19) we may assume that the entries of A are contained in m. Now we define a $g+1$ by $r+s$ matrix

$$B = \left(\begin{array}{ccc} A & | & 0 \\ \hline a_1 & \dots & a_{r+s} \end{array} \right).$$

Then the entries of the matrix $X - B$ are contained in the maximal ideal of S and hence form an S-regular sequence. Modulo this sequence, the S-regular sequence $\varepsilon_1, \dots, \varepsilon_{g+1}$ specializes to the R-regular sequence $\alpha_1, \dots, \alpha_g, b$, and thus again by Lemma 1.9, (S, L_1) is a deformation of (R, H_1'). In particular, $H_1' = (\alpha_1, \dots, \alpha_g, b, d)$ where d is the image of μ in R.

Since $\operatorname{grade}(H_1'/(\alpha_1, \dots, \alpha_g)) = 1$ we may use Lemma 2.9 to conclude that for some $e \in m$, $\alpha_1, \dots, \alpha_g$, $d + eb$ form an R-regular sequence. Now consider the linked R-ideal $H_2' = (\alpha_1, \dots \alpha_g, d + eb) : H_1'$. On the other hand, let Y be another variable, set $T = R[X, Y]_{(m, X, Y)} = S[Y]_{(m, X, Y)}$, and in this ring define the link $L_2 = (\varepsilon_1, \dots, \varepsilon_g, \mu + Y\varepsilon_{g+1})T : L_1 T$ (in the language of [23], L_2 is a general double link of H_0). Then the entries of the matrix $X - B$ together with the element $Y - e$ are contained in the maximal ideal of T and hence form a T-regular sequence. Modulo this sequence, the T-regular sequence $\varepsilon_1, \dots, \varepsilon_g, \mu + Y\varepsilon_{g+1}$ specializes to the R-regular sequence $\alpha_1, \dots, \alpha_g, d + eb$, and the ideal $L_1 T$ specializes to the ideal H_1'. Thus by Lemma 1.9, (T, L_2) is a deformation of (R, H_2'). On the other hand, also the entries of the matrix $X - C$ and the variable Y are contained in the maximal ideal of T and hence form a T-regular sequence. Modulo this sequence, $\varepsilon_1, \dots, \varepsilon_g, \mu + Y\varepsilon_{g+1}$ specializes to the R-regular sequence $\gamma_1, \dots, \gamma_g, \delta$ and $L_1 T$ specializes to H_1. Therefore again by Lemma 1.9, (T, L_2) is a deformation of (R, H_2). Now we have seen that (R, H_2) and (R, H_2') have a common deformation, namely (T, L_2). (This could have also been deduced from the work of Kustin and Miller, [24].) Therefore it suffices to prove that H_2' is a sum of two geometrically linked licci R-ideals.

Thus replacing $H_0 \sim H_1 \sim H_2$ by $H_0 \sim H_1' \sim H_2'$, we are now in the following situation: There is a sequence of links $H_0 \sim H_1 \sim H_2$ such that H_2

is Gorenstein, but not a complete intersection, and moreover

(2.22) $H_0 = I+J$ where I and J are licci R-ideals which are geometrically linked with respect to the regular sequence $\alpha_1, \dots, \alpha_g$,

(2.23) $H_1 = (\alpha_1, \dots, \alpha_g, \beta, \delta)$, where $\alpha_1, \dots, \alpha_g, \beta$ is an R-regular sequence defining the link $H_0 \sim H_1$, and $\alpha_1, \dots, \alpha_g, \delta$ is an R-regular sequence defining the link $H_1 \sim H_2$.

Under these assumptions we will show that H_2 is the sum of two geometrically linked licci R-ideals. This will be the content of Lemma 2.26. Before we can prove Lemma 2.26 however, we need a better description of the ideal H_1:

Lemma 2.24. *Let R be a local Gorenstein ring, let I and J be Cohen-Macaulay R-ideals of* grade g *which are geometrically linked with respect to the regular sequence $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$, set $H = I + J$, let β be an element in H such that $\boldsymbol{\alpha}, \beta$ form a regular sequence, and write $\beta = a + b$, where $a \in I$, $b \in J$.*

Then $(\boldsymbol{\alpha}, \beta) : H = (\boldsymbol{\alpha}, a, b)$.

Proof. We may replace R by $R/(\boldsymbol{\alpha})$ to assume that $I \cap J = 0$, and hence $H = I \oplus J$. We will prove that $(\beta) : H = (a, b)$.

Now R is a Gorenstein ring, I and J are Cohen-Macaulay ideals of grade zero, and β is regular on R. Then Lemma 1.9 implies that

$$(\beta) : (I, \beta) = ((0 : I), \beta) \quad \text{and} \qquad (2.25)$$
$$(\beta) : (J, \beta) = ((0 : J), \beta).$$

Now we are able to compute the ideal $(\beta) : H$.

$$\begin{aligned}
(\beta) : H &= (\beta) : (I + J) \\
&= (\beta) : [(I, \beta) + (J, \beta)] \\
&= [(\beta) : (I, \beta)] \cap [(\beta) : (J, \beta)] \\
&= ((0 : I), \beta) \cap ((0 : J), \beta) \quad \text{(by (2.25))} \\
&= (J, \beta) \cap (I, \beta) \\
&= (J, a) \cap (I, b) \\
&= [(a) \oplus J] \cap [I \oplus (b)] \\
&= (a) \oplus (b) = (a, b). \quad \square
\end{aligned}$$

In the situation of (2.22) and (2.23), Lemma 2.24 implies that $H_1 = (\boldsymbol{\alpha}, \beta, \delta) = (\boldsymbol{\alpha}, a, b)$ where $\beta = a + b$ with $a \in I$, $b \in J$. Thus we do not alter (2.23) by assuming that $\delta = ra + sb$ for $r \in R$, $s \in R$. Since H_2 is Gorenstein and H_1 is not a complete intersection, r and s cannot be both contained in m. Without loss of generality, we may suppose that $r = 1$, and so $\delta = a + sb$. If $s \in m$, we deform δ to $\tilde{\delta} = a + Yb \in R[Y]_{(m,Y)}$ (Y is a variable), specialize $\tilde{\delta}$ to $\delta' = a + s'b$, where $s' \in m$ is regular on $R/(\boldsymbol{\alpha})$, and apply Lemma 1.9 (notice that δ' is automatically regular on $R/(\boldsymbol{\alpha})$, since $\operatorname{Ass}(R/(\boldsymbol{\alpha})) = \operatorname{Ass}(R/I) \cup \operatorname{Ass}(R/J)$, and $\beta = a + b$ is regular on $R/(\boldsymbol{\alpha})$, and $a \in I$, $b \in J$).

Then we may assume that $\boldsymbol{\alpha}, s$ form an R-regular sequence in case $s \in m$. In either case, using the equation $\beta = a + b$, we obtain

$$\begin{aligned}\delta &= a + sb \\ &= \beta + c \qquad (c \in J) \\ &= s\beta + d \qquad (d \in I),\end{aligned}$$

where either s is a unit or else $\boldsymbol{\alpha}, s$ form an R-regular sequence. Now Theorem 2.17 follows from Lemma 2.26:

Lemma 2.26. *Let R be a local Gorenstein ring, let I and J be Cohen-Macaulay R-ideals of* grade g *which are geometrically linked with respect to the regular sequence $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$, set $H_0 = I + J$, and consider a sequence of links $H_0 \sim H_1 \sim H_2$. We further assume that the link $H_0 \sim H_1$ is defined by the regular sequence $\boldsymbol{\alpha}, \beta$, and that the link $H_1 \sim H_2$ is defined by the regular sequence $\boldsymbol{\alpha}, \delta$ with $\delta = \beta + c = s\beta + d$, where $c \in J, d \in I$, and either s is a unit or $\boldsymbol{\alpha}, s$ form an R-regular sequence.*

Then $H_2 = I + J'$, where $J' = (\alpha_1, \dots, \alpha_{g-1}, sJ)$, and the ideals I and J' are geometrically linked with respect to the regular sequence $\alpha_1, \dots, \alpha_{g-1}, s\alpha_g$. If I and J are licci, then obviously the same holds true for I and J'.

Proof. We first note that $\alpha_1, \dots, \alpha_{g-1}, s\alpha_g$ form an R-regular sequence, since either s is a unit, or else $\alpha_1, \dots, \alpha_{g-1}, s$ are a regular sequence. But then by [5, 6.1.11],

$$\begin{aligned}(\alpha_1, \dots, \alpha_{g-1}, s\alpha_g) : I &= (\alpha_1, \dots, \alpha_{g-1}, s\alpha_g, s[(\alpha_1, \dots, \alpha_g) : I]) \\ &= (\alpha_1, \dots, \alpha_{g-1}, sJ) = J',\end{aligned}$$

which proves that I and J' are linked with respect to $\alpha_1, \dots, \alpha_{g-1}, s\alpha_g$. To show that this link is geometric, we simply observe that

$$\operatorname{grade}(I + J') = \operatorname{grade}(I + sJ) > \operatorname{grade} I,$$

where the latter inequality holds since $\operatorname{grade}(I + J) > \operatorname{grade} I$, and since by assumption, s is not contained in any associated prime of I.

It remains to prove that $H_2 = I + J'$. Since both ideals contain $(\alpha_1, \dots, \alpha_g)$, we may factor out $(\alpha_1, \dots, \alpha_g)$ to assume that $IJ = 0$ and $\operatorname{grade} H_2 = 1$. Then $H_2 = (\delta) : H_1 = \delta(H_1)^{-1}$, where $(H_1)^{-1}$ denotes the inverse fractional ideal of H_1 in the total ring of quotients of R. Likewise, $H_0 = (\beta) : H_1 = \beta(H_1)^{-1}$. Combining the two equations we obtain that $H_2 = \delta\beta^{-1}H_0$, and therefore

$$H_2 = \delta\beta^{-1}(I + J) = \beta^{-1}(\delta I + \delta J). \tag{2.27}$$

However $\delta = \beta + c = s\beta + d$ where $c \in J$, $d \in I$, and $cI + dJ \subset IJ = 0$. Thus (2.27) implies that

$$\begin{aligned}H_2 &= \beta^{-1}(\beta I + cI + s\beta J + dJ) \\ &= \beta^{-1}(\beta I + \beta sJ) = I + sJ = I + J'. \quad \square\end{aligned}$$

Now the proof of Theorem 2.17 is complete. Combining Theorem 2.1, Lemma 1.11, and Theorem 2.17, we obtain the following characterization of licci Gorenstein ideals:

Corollary 2.28. *Let* R *be a local Gorenstein ring with infinite residue class field and let* K *be an* R*-ideal with* $\operatorname{grade} K \geq 2$.

Then K *is a licci Gorenstein ideal if and only if there exist geometrically linked licci* R*-ideals* I *and* J *such that* (R, K) *is a generalized localization of* $(R, I+J)$.

3. The depth of the Koszul homology

In this and the next sections, depth conditions on the Koszul homology will play an important role. Let I be an ideal in a local Gorenstein ring R, then by $H_i(I)$ we denote the ith homology of the Koszul complex for some fixed generating set of I. Now $H_i(I)$ is an R/I-module with $\dim H_i(I) = \dim R/I$ or $H_i(I) = 0$ [16, Remark 1.5], and for a fixed integer t, the property of $H_i(I)$ being Cohen-Macaulay for all $0 \leq i \leq t$ does not depend on the chosen generating set of I [13, 1.5]. The ideal I is called *strongly Cohen-Macaulay* if $H_i(I)$ are Cohen-Macaulay for all i [14]. This notion has turned out to be very useful in the study of blowing-up rings and residual intersections ([12], [14], [16], [21], [30], [31]). Huneke showed that the strong Cohen-Macaulay property is preserved under even linkage ([13, 1.12], cf. also [15]), and later, Vasconcelos proved that for perfect ideals of grade 3, the Cohen-Macaulayness of the first Koszul homology is invariant under even and odd linkage [34, 2.4]. Although the latter result cannot be generalized to higher grade, one can still ask how the depth of the first Koszul homology of a Cohen-Macaulay ideal I is reflected in properties of an ideal J belonging to the odd linkage class of I. This question is answered in Theorem 3.1, where we show that the depth of $H_1(I)$ coincides with the depth of $S_2(\omega_{R/J})$, the second symmetric power of the canonical module of R/J (note that the depth of $H_1(I)$ does not depend on the chosen generating set of I, cf. [13, 1.2]). We then use Theorem 3.1 to explain how depth conditions on the first Koszul homology pass to the sum of two geometrically linked Cohen-Macaulay ideals. This is done in Corollary 3.10 and Theorem 3.11, which also yield a criterion for when the Cohen-Macaulayness of the first Koszul homology is preserved under odd linkage. Besides Cohen-Macaulayness another depth condition on the first Koszul homology comes into play: An ideal I is called *syzygetic* if the natural epimorphism $S_2(I) \to I^2$ is injective, or equivalently, if the first map in the standad exact sequence

$$H_1(I) \to \bigoplus R/I \to I/I^2 \to 0$$

is injective [30, 1.2]. It is clear from the latter description that if I is generically a complete intersection, then I is syzygetic if and only if $H_1(I)$ is R/I-torsion free.

To formulate our results, we also need to consider depth conditions on the conormal module: A Cohen-Macaulay ideal I is called *strongly nonobstructed* if the twisted conormal module of I, $I \otimes_R \omega_{R/I}$, is a Cohen-Macaulay R/I-module. Due to the work of Herzog, this notion plays an important role in deformation theory [10]. In fact, if I is strongly nonobstructed and R is a power series ring over a field k and R/I is reduced, then there are no obstructions for lifting infinitesimal deformations of R over k [10, 1.4, cf. also 2.3]. Buchweitz has shown that the property of being strongly nonobstructed is preserved under linkage (at least if R contains a field and all ideals are generically complete intersectons, [5, 6.2.11]), and later it was proved in [6] that even the depth of the twisted conormal module is an invariant of the linkage class. Of course any licci ideal is strongly nonobstructed as well as strongly Cohen-Macaulay by the work of Buchweitz and Huneke. In the last section of this paper however, we will show that neither property (even if the strong Cohen-Macaulayness is required for the entire linkage class) characterizes licci ideals. To accomplish this, we will need Corollary 3.10, which says that if I and J are two geometrically linked Cohen-Macaulay ideals and the twisted conormal module of I is R/I-torsion free and I as well as J are syzygetic, then $I+J$ is syzygetic. This fact can be used to show that a given twisted conormal module has nontrivial torsion, since computationally it is much easier to check the syzygetic property of ideals than torsion freeness of a twisted conormal module.

Our Theorem 3.1 was inspired by Vasconcelos' paper [34], where the same result is shown for ideals of grade three. I am also very grateful to Wolmer Vasconcelos for simplifying my original proof of Theorem 3.1.

Theorem 3.1. *Let R be a local Gorenstein ring, and consider a link $I \sim J$ of Cohen-Macaulay R-ideals.*

Then $\operatorname{depth} H_1(I) = \operatorname{depth} S_2(\omega_{R/J})$.

Proof. Let $d = \dim R$ and $g = \operatorname{grade} I$. After adjoining a variable X to R and replacing I by the ideal $(I, X)R[X]_{(m,X)}$ (where m is the maximal ideal of R), we may assume that $g > 0$ (by [13, 1.4] the depth of the first Koszul homology is invariant under this operation).

We first show the existence of an exact sequence

$$0 \to M \to S_2(I) \to S_2(\omega_{R/J}) \to 0 \tag{3.2}$$

where $\operatorname{depth} M = d - g + 1$ (cf. also [34, proof of 2.4]).

Let $\alpha_1, \dots, \alpha_g$ be a regular sequence defining the link $I \sim J$, and let $L = (\alpha_1, \dots, \alpha_g)$. Then the exact sequence from Proposition 1.2(c),

$$0 \to L \to I \to \omega_{R/J} \to 0$$

induces an exact sequence

$$0 \to L \cdot I \to S_2(I) \to S_2(\omega_{R/J}) \to 0,$$

where $L \cdot T$ denotes the image of $L \otimes_R I$ in $S_2(I)$. Now (3.2) follows once we have shown that $\operatorname{depth} L \cdot I = d - g + 1$. To this end, we first claim that

$L \cdot I \cong LI$, where LI is the ideal product of L and I in R. There is a commutative diagram of natural maps

$$\begin{array}{ccc} L \cdot I & \xrightarrow{\varphi} & LI \\ {\scriptstyle\psi}\uparrow & & \uparrow \\ S_2(L) & \xrightarrow[\sim]{\varepsilon} & L^2 \end{array}$$

where ε is an isomorphism since L is a complete intersection. It will follow that φ is injective once we have shown that $\ker\varphi \subset \operatorname{im}\psi$. So let $y \in \ker\varphi$, and write $y = \sum_{i=1}^{g} \alpha_i \cdot h_i$ with $h_i \in I$. Then $\sum_{i=1}^{g} \alpha_i h_i = 0$ in R, and since $\alpha_1, \dots, \alpha_g$ form an R-sequence, we conclude that $h_i \in (\alpha_1, \dots, \alpha_g) = L$. But then $y \in \operatorname{im}\psi$. Thus we have shown that $L \cdot I \cong LI$.

To see that $\operatorname{depth} LI = d - g + 1$, we simply consider the exact sequence

$$0 \to LI \to L \to L/LI \to 0,$$

and notice that

$$\begin{aligned} L/LI &\cong L \otimes_R R/I \cong (L/L^2) \otimes_{R/L} R/I \\ &\cong (R/L)^g \otimes_{R/L} R/I \cong (R/I)^g . \end{aligned}$$

Thus $\operatorname{depth} LI = d - g + 1$, and we have proven (3.2).

Now fix a generating set $f_1, \dots, f_n$ of I (by [13, 1.2], $\operatorname{depth} H_1(I)$ is independent of this choice), and let B_1, Z_1, K_1 be the first boundaries, cycles, and chains of the Koszul complex of $f_1, \dots, f_n$. We will show that there is an exact sequence

(3.3) $$0 \to B_1 \to \bigoplus^n I \to S_2(I) \to 0$$

(this is essentially in [30]).

To this end consider the exact sequence

$$0 \to Z_1 \to K_1 = \bigoplus_{i=1}^{n} Re_i \xrightarrow{\Pi} I \to 0$$

where $\Pi(e_i) = f_i$. Tensoring by $I \otimes_R$, we obtain an exact sequence

$$0 \to IZ_1 \to \bigoplus_{i=1}^{n} Ie_i \xrightarrow{\mathrm{id}_I \otimes \Pi} I \otimes_R I \to 0.$$

Now

$$IZ_1 \subset B_1 = (f_i e_j - f_j e_i \mid 1 \le i < j \le n) \subset \bigoplus_{i=1}^{n} Ie_i ,$$

and hence the above exact sequence yields

$$0 \to B_1 \to \bigoplus_{i=1}^{n} Ie_i \to I \otimes_R I/N \to 0$$

where $N = (f_i \otimes f_j - f_j \otimes f_i | 1 \le i < j \le n)$. On the other hand, $I \otimes_R I/N \cong S_2(I)$. This completes the proof of (3.3).

Now Theorem 3.1 follows easily from (3.2), (3.3), and the fact that $\operatorname{depth} H_1(I) = \operatorname{depth} B_1 - 1$. If $\operatorname{depth} S_2(I) \ge d - g + 1$, then $\operatorname{depth} S_2(\omega_{R/J}) \ge d - g$ and $\operatorname{depth} H_1(I) \ge d - g$. But then $\operatorname{depth} S_2(\omega_{R/J}) = d - g = \operatorname{depth} H_1(I)$. If however $\operatorname{depth} S_2(I) \le d - g$, then

$$\operatorname{depth} S_2(\omega_{R/J}) = \operatorname{depth} S_2(I) = \operatorname{depth} H_1(I). \quad \square$$

The next two corollaries follow immediately from Theorem 3.1.

Corollary 3.4. *Let R be a local Gorenstein ring, and let I and J be two Cohen-Macaulay R-ideals in the same odd linkage class, then*

(a) $\operatorname{depth} H_1(I) = \operatorname{depth} S_2(\omega_{R/J})$,
(b) *$H_1(I)$ satisfies Serre's condition (S_k) if and only if $S_2(\omega_{R/J})$ satisfies (S_k).*

Corollary 3.5. *Let R be a local Gorenstein ring, and let I and J be two Cohen-Macaulay R-ideals in the same even linkage class, then*

(a) $\operatorname{depth} H_1(I) = \operatorname{depth} H_1(J)$ (*cf. also* [13, *proof of* 1.11]),
(b) $\operatorname{depth} S_2(\omega_{R/I}) = \operatorname{depth} S_2(\omega_{R/J})$.

From Theorem 3.1 we also obtain the following result of Vasconcelos [34, 2.8].

Corollary 3.6. *Let R be a local Gorenstein ring, and let I be a Cohen-Macaulay R-ideal such that $r(R/I) = 2$.*

Then $S_2(\omega_{R/I})$ is Cohen-Macaulay.

Proof. Consider a link $I \sim J$. Then by Proposition 1.2(c), $d(J) \le 2$, and thus by [3], $H_1(J)$ is Cohen-Macaulay. Now the claim follows from Theorem 3.1. $\square$

Before we prove our next theorem, another lemma is needed.

Lemma 3.7. *Let R be a local Gorenstein ring, let I and J be two geometrically linked Cohen-Macaulay R-ideals, and write $K = I + J$. Further assume that I is syzygetic and that $I \otimes_R \omega$ is torsion free as an R/I-module where $\omega = \omega_{R/I}$.*

Then there are exact sequences

(a) $0 \to I^2 \to IK \to I \otimes_R \omega \to 0$,
(b) $0 \to IK \to S_2(K) \to S_2(\omega) \to 0$ *with* $\operatorname{depth} S_2(\omega) = \operatorname{depth} H_1(J)$.

Proof. Let $\alpha = \alpha_1, \dots, \alpha_g$ be a regular sequence defining the link $I \sim J$. Then $(\alpha) = I \cap J$ since the link $I \sim J$ is geometric. Thus by Proposition 1.2(c), $\omega \cong J/(\alpha) = J/I \cap J \cong K/I$, and we have an exact sequence

$$0 \to I \to K \to \omega \to 0.$$

Tensoring by $I\otimes_R$, yields a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
\wedge^2 I & = & \wedge I^2 & & & & \\
\downarrow & & \downarrow\rho & & & & \\
I\otimes_R I & \longrightarrow & I\otimes_R K & \longrightarrow & I\otimes_R \omega & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
S_2(I) & & I\otimes_R K/\rho(\wedge^2 I) & & & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

Now $S_2(I)\cong I^2$ since I is syzygetic, and from the Snake Lemma we obtain an exact sequence

$$I^2 \xrightarrow{\psi} I\otimes_R K/\rho(\wedge^2 I) \to I\otimes_R \omega \to 0 .$$

Notice that for all $p\in \operatorname{Spec}(R)$ with $K\not\subset p$, $(I\otimes_R K/\rho(\wedge^2 I))_p \cong I_p$ and ψ_p is the natural embedding $I_p^2 \to I_p$. In particular, $\ker\psi$ is a R-torsion submodule of I^2 and hence zero. Thus we have an exact sequence

$$0\to I^2 \to I\otimes_R K/\rho(\wedge^2 I) \to I\otimes_R \omega \to 0 . \tag{3.8}$$

Now part (a) follows from (3.8) once we have shown that

$$I\otimes_R K/\rho(\wedge^2 I) \cong IK . \tag{3.9}$$

To prove (3.9), consider the natural projection

$$\Pi\colon I\otimes_R K/\rho(\wedge^2 I) \twoheadrightarrow IK .$$

By the above, Π_p is injective for all $p\in\operatorname{Spec}(R)$ with $K\not\subset p$. Suppose that $\ker\Pi\neq 0$, and let p be a minimal element in $\operatorname{Supp}(\ker\Pi)$. Then $K\subset p$, and hence by the R/I-torsion freeness of $I\otimes_R\omega$, $\operatorname{depth}(I\otimes_R \omega)_p\geq 1$. But then by (3.8), $\operatorname{depth}(I\otimes_R K/\rho(\wedge^2 I))_p \geq 1$, and hence $\operatorname{depth}(\ker\Pi)_p\geq 1$, which is impossible by the minimality of p. This contradiction proves (3.9).

We now prove part (b). Taking second symmetric powers in the exact sequence $0\to I\to K\to\omega\to 0$, we obtain an exact sequence

$$I\otimes_R K/\rho(\wedge^2 I)\to S_2(K)\to S_2(\omega)\to 0 .$$

Then (3.9) implies that

$$IK \xrightarrow{\chi} S_2(K) \longrightarrow S_2(\omega) \longrightarrow 0$$

is exact. It is easy to see that $\ker\chi$ is an R-torsion module and hence zero. Also notice that by Theorem 3.1, $\operatorname{depth} S_2(\omega) = \operatorname{depth} H_1(J)$. $\square$

Corollary 3.10. *If with the assumptions of Lemma* 3.7, J *is syzygetic, then also* K *is syzygetic.*

Proof. Suppose that K is not syzygetic. Then $\delta(K) \neq 0$, where $\delta(K)$ is the kernel of the natural map from $S_2(K)$ to K^2. We may localize at a minimal prime ideal in $\mathrm{Supp}(\delta(K))$ to assume that $\operatorname{depth} \delta(K) = 0$. Notice that K is still a proper ideal in $R = R_p$, and in particular, $\dim R/J \geq 1$. Since $\delta(K) \subset S_2(K)$ and $\operatorname{depth} \delta(K) = 0$, it follows that $\operatorname{depth} S_2(K) = 0$. But then by Lemma 3.7(b), $\operatorname{depth} H_1(J) = 0$. On the other hand, J is syzygetic, and hence $H_1(J)$ is a submodule of a free R/J-module where $\operatorname{depth} R/J \geq 1$. Thus $\operatorname{depth} H_1(J) \geq 1$, which yields a contradiction. □

Theorem 3.11. *Let* R *be a local Gorenstein ring, let* I *and* J *be two geometrically linked Cohen-Macaulay* R*-ideals, and write* $K = I + J$. *Further assume that* I *satisfies* (CI_1) *and that the* R/I*-modules* $H_1(I)$ *and* $I \otimes_R \omega_{R/I}$ *satisfy Serre's condition* (S_2).

(a) *Then the following are equivalent:*

(i) J *is syzygetic,*

(ii) K *is syzygetic.*

(b) *If in addition,* I *is strongly nonobstructed, then the following are equivalent:*

(i) $H_1(I)$ *and* $H_1(J)$ *are Cohen-Macaulay,*

(ii) K *is syzygetic and strongly nonobstructed.*

Proof. First notice that the assumptions of Lemma 3.7 are satisfied.

We now prove part (a). By Corollary 3.10 it suffices to show that (ii) implies (i). Since the link $I \sim J$ is geometric and hence J is generically a complete intersection, it suffices to prove that $H_1(J)$ is torsion free. Let $p \in V(J)$. If $I \not\subset p$ or if $\dim(R/J)_p \leq 1$, then J_p is an almost complete intersection and it is well known that $H_1(J_p)$ is torsion free. Thus after localizing at p, we may assume that I and J are still linked and that $\dim R/I = \dim R/J \geq 2$. We need to show that $\operatorname{depth} H_1(J) \geq 1$. However, by our assumption on I, $\operatorname{depth} H_1(I) \geq 2$ and $\operatorname{depth} I \otimes_R \omega_{R/I} \geq 2$. Hence the standard exact sequence

$$0 \to H_1(I) \to \bigoplus R/I \to I/I^2 \to 0 \tag{3.12}$$

and the exact sequence in Lemma 3.7(a) imply that $\operatorname{depth} IK \geq 2$. Moreover, $\operatorname{depth} S_2(K) \geq 1$ since we assumed K to be syzygetic, and hence by Lemma 3.7(b), $\operatorname{depth} H_1(J) = \operatorname{depth} S_2(\omega_{R/I}) \geq 1$.

We now prove part (b). Write $d = \dim R$ and $g = \operatorname{grade} I$. We first show that (i) implies (ii). By Corollary 3.10, K is syzygetic. Moreover, by (3.12) and Lemma 3.7, $\operatorname{depth} S_2(K) \geq d - g$. Since $S_2(K) \cong K^2$, we conclude that $\operatorname{depth} K/K^2 \geq d - g - 1$. But then K is strongly nonobstructed since K is a Gorenstein ideal of grade $g + 1$ (Proposition 1.3).

Next we show that (ii) implies (i). By part (a), J is syzygetic and by [6], $J \otimes_R \omega_{R/J}$ is Cohen-Macaulay. Now both ideals have this property, and the problem is symmetric in I and J (except that $H_1(J)$ need not be (S_2), which is irrelevant for the remainder of the proof). Assume (ii) and suppose that (i) does not hold. Then we may assume that $H_1(I)$ is not Cohen-Macaulay and $\operatorname{depth} H_1(I) \leq \operatorname{depth} H_1(J)$. Now $\operatorname{depth} H_1(I) \leq d-g-1$, $\operatorname{depth} I \otimes_R \omega_{R/I} = d-g$, and by (ii), $\operatorname{depth} S_2(K) = d-g$ (note that K is a Gorenstein ideal of grade $g+1$). But then (3.12) and the exact sequences in Lemma 3.7 imply that $\operatorname{depth} S_2(\omega_{R/I}) = \operatorname{depth} H_1(I) - 1$, and hence $\operatorname{depth} H_1(J) < \operatorname{depth} H_1(I)$, which is a contradiction to the above assumption. □

Notice that in Theoem 3.11(b), $H_1(K)$ is Cohen-Macaulay, because K is Gorenstein and hence $H_1(K)$ is a syzygy module of a maximal Cohen-Macaulay module.

4. Sums of doubly linked Gorenstein ideals

In the preceding two sections we studied sums of Cohen-Macaulay ideals of grade g, I and J, which are directly linked. In order to obtain any results we had to assume that $I+J$ had grade at least $g+1$, $g+1$ being the "generic" grade for ideals of this type. We now take these investigations one step further and consider sums of Gorenstein ideals of grade g, I_0 and I_2, which are linked in two steps. In Proposition 2.16 we had seen that either I_0 and I_2 become equal after deformation or else there exists a tight double link joining I_0 and I_2. Thus it seems natural to assume that the link $I_0 \sim I_1 \sim I_2$ is always tight. We also assume that I_0+I_2 has grade at least $g+2$, $g+2$ being the "generic" grade for ideals of this type (cf. Lemma 4.1, Remark 4.2). We then obtain positive results concerning the Cohen-Macaulayness of I_0+I_2 (Theorem 4.3), and we can compute the type of R/I_0+I_2 (Remark 4.5). However, the example of (5.1) will illustrate that I_0+I_2 need not be licci, even if I_0 (and I_2) has this property.

We begin by establishing that $g+2$ is the "generic" grade of I_0+I_2.

Lemma 4.1. *Let R be a regular local ring, and let $I_0 \sim I_1 \sim I_2$ be a tight double link with I_0 and I_2 Gorenstein ideals of* grade $g>0$.

Then every minimal prime ideal of I_0+I_2 has height at most $g+2$.

Proof. Let $\beta, \delta, \boldsymbol{\alpha} = \alpha_1, \dots, \alpha_{g-1}$ be elements in R such that $\boldsymbol{\alpha}, \beta$, and $\boldsymbol{\alpha}, \delta$ form R-regular sequences contained in I_0 and I_2 respectively, and $I_1 = (\boldsymbol{\alpha}, \beta, \delta) = (\boldsymbol{\alpha}, \beta) : I_0 = (\boldsymbol{\alpha}, \delta) : I_2$. After factoring out the ideal $(\boldsymbol{\alpha})$ we are in the situation where $g=1$ and R is a complete intersection.

Let I_1^{-1} be the inverse fractional ideal of I_1 in the total quotient ring of R. Then $I_0 = (\beta) : I_1 = \beta I_1^{-1}$, and $I_2 = (\delta) : I_1 = \delta I_1^{-1}$. Therefore $I_0 + I_2 = (\beta, \delta) I_1^{-1} = I_1 I_1^{-1}$. Now let p be a minimal prime of I_0+I_2. Since $I_0 + I_2 = I_1 I_1^{-1}$, it follows that the ideal $(I_1)_p$ is not invertible, but that $(I_1)_q$ is invertible for all q in the punctured spectrum of R_p. Moreover, I_1 is an unmixed ideal of grade one in a complete intersection ring R. In this situation, [20, 2.2] implies that $\dim R_p \leq 3$. □

Remark 4.2. Let R' be a local Gorenstein ring, let $I' = (f_1, \dots, f_n)$ be a Gorenstein R'-ideal of grade $g > 0$, let X be a finite set of variables over R', $p \in \operatorname{Spec}(R'[X])$, $R = R'[X]_p$, and let $I_0 = I'R \sim I_1 \sim I_2$ be a tight double link in R. Assume that I' satisfies (CI_1) and that the regular sequence $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_{g-1}$ in Definition 1.5 is of the form

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{g-1} \end{pmatrix} = Y \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

where Y is a $g-1$ by n matrix whose entries are linearly independent linear forms in $R'[X]$ (this is always satisfied if the tight double link is semigeneric in the sense of [24]).

Then $\operatorname{grade}(I_0 + I_2) \geq g + 2$.

Proof. From the proof of Lemma 4.1 it follows that $\operatorname{grade}(I_0 + I_2) \geq g + 2$ if and only if the $R/(\boldsymbol{\alpha})$-ideal $I_1/(\boldsymbol{\alpha})$ satisfies (CI_1), which in turn is equivalent to $I_0/(\boldsymbol{\alpha})$ being (CI_1) since the two ideals are linked and have grade one.

To verify that $I_0/(\boldsymbol{\alpha})$ satisfies (CI_1) is suffices to show that for every $q \in \operatorname{Spec}(R'[X])$ with $I' \subset q$ and $\dim R'[X]_q \leq g + 1$, the module $(I'R'[X]/(\boldsymbol{\alpha}))_q$ is cyclic. Localizing at $q \cap R'$, we may assume that $q \cap R' = m'$, the maximal ideal of R'. Then $g \leq \dim R' \leq g + 1$, and since I' satisfies (CI_1), we may assume that I' is minimally generated by $f_1, \dots, f_g$. After modifying Y, we obtain that

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{g-1} \end{pmatrix} = Y \begin{pmatrix} f_1 \\ \vdots \\ f_g \end{pmatrix}$$

with Y a $g-1$ by g matrix whose entries are still R'-linearly independent linear forms in $R'[X]$. Thus $\operatorname{ht}(m', I_{g-1}(Y))R'[X] = \dim R' + 2 \geq g + 2 > \operatorname{ht} q$, and since $m' \subset q$, we conclude that $I_{g-1}(Y) \not\subset q$. But then $(I'R'[X]/(\boldsymbol{\alpha}))_q = ((f_1, \dots, f_g)/(\alpha_1, \dots, \alpha_{g-1}))_q$ has to be cyclic. □

Lemma 4.1 and Remark 4.2 motivate the grade assumption in the next theorem, which is the main result of this section.

Theorem 4.3. *Let R be a local Gorenstein ring, let $I_0 \sim I_1 \sim I_2$ be a tight double link with I_0 and I_2 Gorenstein ideals of* grade $g > 0$ (*by Proposition* 2.16, *the double link $I_0 \sim I_1 \sim I_2$ is automatically tight if (R, I_0) and (R, I_2) have no common deformation), and assume that* $\operatorname{grade}(I_0 + I_2) \geq g + 2$.

If $H_1(I_0)$ is Cohen-Macaulay, then $I_0 + I_2$ is a Cohen-Macaulay ideal of grade $g + 2$.

Proof. As in the proof of Lemma 4.1 we reduce to the case $g = 1$. [13, 1.6], guarantees that this reduction does not alter any of our assumptions and conclusions. Set $d = \dim R$. From now on we will assume that $H_1(I_0)$ is Cohen-Macaulay.

First notice that by the grade assumption on $I_0 + I_2$ and the proofs of Lemma 4.1 and Remark 4.2, the ideal I_0 satisfies (CI_1). This fact, the Cohen-Macaulayness of $H_1(I_0)$, and the standard exact sequence (3.12) already imply that I_0^2 is reflexive and $\operatorname{depth} I_0^2 \geq d-1$.

We now claim that $I_0 \cap I_2 = I_0 I_2$. Since clearly $I_0 I_2 \subset I_0 \cap I_2$, it suffices to show that equality holds locally at all associated primes of $I_0 I_2$. However, $I_0 \sim I_2 \sim I_2$ is a double link of ideals of grade one, hence $I_0 \cong I_1^{-1} \cong I_2$, and therefore $I_0 I_2 \cong I_0^2$. We have seen that the latter ideal is reflexive, thus $I_0 I_2$ is reflexive and hence unmixed. Then it suffices to prove that $(I_0 \cap I_2)_q = (I_0 I_2)_q$ for all $q \in V(I_0 I_2)$ with $\dim R_q \leq 1$. But for any such q, either $(I_0)_q = R_q$ or $(I_2)_q = R_q$, and the above equality is obviously true. Thus we have shown that $I_0 \cap I_2 = I_0 I_2$.

Now the standard exact sequence

$$0 \to I_0 \cap I_2 \to I_0 \oplus I_2 \to I_0 + I_2 \to 0$$

together with the isomorphisms $I_0 \cap I_2 \cong I_0 I_2 \cong I_0^2$ and $I_2 \cong I_0$, yields an exact sequence

$$0 \to I_0^2 \to I_0 \oplus I_0 \to I_0 + I_2 \to 0. \tag{4.4}$$

Since $\operatorname{depth} I_0^2 \geq d-1$, we obtain from (4.4) that $\operatorname{depth} R/I_0 + I_2 \geq d-3$. Since on the other hand, $\operatorname{grade}(I_0 + I_2) \geq g+2 = 3$, we conclude that $I_0 + I_2$ is a Cohen-Macaulay ideal of grade $3 = g+2$. □

In the next remark, we give an estimate for the type of the Cohen-Macaulay ring $R/I_0 + I_2$ from Theorem 4.3. First we need a definition: Let $\alpha = \alpha_1, \ldots, \alpha_{g-1}$ be a regular sequence as in Definition 1.5, then by t, $0 \leq t \leq g-1$, we denote the maximal number of elements $\gamma = \gamma_1, \ldots, \gamma_t$ such that γ form part of a minimal generating set of the ideal (α), and γ "split I_0 as hypersurface sections" which means that there is a Cohen-Macaulay ideal J of grade $g-t$ such that $I_0 = (J, \gamma)$.

Remark 4.5. In addition to the assumptions of Theorem 4.3, let R be a power series ring over a field and assume that I_0 is strongly nonobstructed.

Then $r(R/I_0 + I_2) = g - 1 - t$. If I_0 or I_2 or $I_0 + I_2$ is not a complete intersection, then $2 \leq r(R/I_0 + I_2) \leq g-1$.

Proof. With the notations preceding the remark, let "$\overline{}$" denote reduction modulo $(\alpha) = (\alpha_1, \ldots, \alpha_{g-1})$, and set $S = R/I_0 = \overline{R}/\overline{I}_0$. Then there is an exact sequence

$$0 \to S^{g-1} \to I_0/I_0^2 \to \overline{I}_0/\overline{I}_0^2 \to 0$$

which induces an exact sequence

$$\operatorname{Hom}_S(I_0/I_0^2, S) \to \operatorname{Hom}_S(S^{g-1}, S) \to \operatorname{Ext}^1_S(\overline{I}_0/\overline{I}_0^2, S) \to 0,$$

since by assumption, $\operatorname{Ext}^1_S(I_0/I_0^2, S) \cong \operatorname{Ext}^1_S(I_0/I_0^2 \otimes_S \omega_S, \omega_S) = 0$ (cf. [10, 1.2]). From this exact sequence one easily sees that $g - 1 - \nu(\operatorname{Ext}^1_S(\overline{I}_0/\overline{I}_0^2, S))$ is equal to the maximal rank of a free direct summand of I_0/I_0^2 which is contained in $S^{g-1} = \bigoplus_{i=1}^{g-1} S(\alpha_i + I_0^2)$. But on the other hand, this rank is t (cf. [11, proof of 1.3]).

Thus to show that $r(R/I_0 + I_2) = g - 1 - t$, we have to prove that

$$r(R/I_0 + I_2) = \nu(\operatorname{Ext}^1_S(\overline{I}_0/\overline{I}_0^2, S)). \tag{4.6}$$

To this end consider the exact sequence from (4.4),

$$0 \to \overline{I}_0^2 \to \overline{I}_0 \oplus \overline{I}_0 \to \overline{R} \to \overline{R}/\overline{I_0 + I_2} \to 0.$$

Then we obtain isomorphisms

$$\omega_{R/I_0+I_2} \cong \operatorname{Ext}^3_{\overline{R}}(\overline{R}/\overline{I_0+I_2}, \overline{R}) \cong \operatorname{Ext}^1_{\overline{R}}(\overline{I}_0^2, \overline{R}) \cong \operatorname{Ext}^2_{\overline{R}}(\overline{I}_0/\overline{I}_0^2, \overline{R}).$$

On the other hand by local duality, $\operatorname{Ext}^2_{\overline{R}}(\overline{I}_0/\overline{I}_0^2, \overline{R}) \cong \operatorname{Ext}^1_S(\overline{I}_0/\overline{I}_0^2, S)$. Therefore $\omega_{R/I_0+I_2} \cong \operatorname{Ext}^1_S(\overline{I}_0/\overline{I}_0^2, S)$, which proves (4.6).

It remains to show that I_0+I_2 is not Gorenstein in case I_0, I_2, or I_0+I_2 is not a complete intersection. Suppose, I_0+I_2 is Gorenstein. Then $t = g-2$ and hence $I_0 = (J, \gamma_1, \dots, \gamma_{g-2})$, where J is a perfect Gorenstein ideal of grade 2. Thus J would have to be a complete intersection [29]. Let "$'$" denote reduction modulo $(\gamma_1, \dots, \gamma_{g-2})$. Then $I_0' = J'$ is a complete intersection of grade 2 in R', and considering the link $I_0' \sim I_1' \sim I_2'$ we conclude that the R'-ideal I_2' is licci and hence perfect. Thus I_2' is a complete intersection, since this ideal is also Gorenstein of grade 2. Therefore $\nu(I_0' + I_2') \leq 4 = \operatorname{grade}(I_0' + I_2')$. Thus I_0, I_2, and $I_0 + I_2$ would have to be complete intersections in R. □

5. Examples

In this section we use the results of the previous two sections to construct two examples. The first example is a perfect prime ideal which is not strongly nonobstructed, but whose entire linkage class is strongly Cohen-Macaulay. Conversely, the second example is a perfect prime ideal which is strongly nonobstructed, but not strongly Cohen-Macaulay. The first ideal is a sum of two tightly linked Gorenstein ideals of grade 3, whereas the second ideal is essentially obtained from the first one as a sum of two geometrically linked ideals. Since by [5] and [13], any licci ideal is strongly nonobstructed and strongly Cohen-Macaulay, it follows that neither of the above ideals is licci. In particular our examples show that a perfect prime ideal need not be licci even if it is strongly nonobstructed or its entire linkage class is strongly Cohen-Macaulay.

We first have to fix some notation. Let k be a field of characteristic zero, let $X = (x_{ij})$ be a generic alternating 6 by 6 matrix over k (if we wish we may set $x_{56} = 0$), let $\overline{X}$ be the matrix obtained from X by replacing x_{12} by 0, and set $R = k[[X]]$ (or $k[X]$). For $1 \leq i < j \leq 6$ let $(-1)^{i+j+1} f_{ij} = (-1)^{i+j} f_{ji}$

or $(-1)^{i+j+1}\overline{f}_{ij} = (-1)^{i+j}\overline{f}_{ji}$ be the 4 by 4 Pfaffian obtained from X or $\overline{X}$ by deleting the ith and jth rows and columns. Now we consider the R-ideals

$$I = (f_{i5}, f_{i6}, f_{56} | 1 \leq i \leq 4) \tag{5.1}$$

and

$$\begin{aligned} K &= I_2 \begin{pmatrix} x_{13} & x_{14} & x_{15} & x_{16} \\ x_{23} & x_{24} & x_{25} & x_{26} \end{pmatrix} + (\overline{f}_{15}, \overline{f}_{16}, \overline{f}_{25}, \overline{f}_{26}) \\ &= I_2 \begin{pmatrix} x_{13} & x_{14} & x_{15} & x_{16} \\ x_{23} & x_{24} & x_{25} & x_{26} \end{pmatrix} + (\overline{f}_{i5}, \overline{f}_{i6}, \overline{f}_{56} | 1 \leq i \leq 4). \end{aligned} \tag{5.2}$$

Here I is simply the ideal generated by all 4 by 4 Pfaffians obtained from X by deleting at least one of the last two rows and columns, and K is the ideal generated by the maximal minors of the matrix consisting of the first two rows of $\overline{X}$ and by the 4 by 4 Pfaffians of $\overline{X}$ which are obtained by deleting at least one of the last two rows and columns.

We are now ready to list the main properties of the ideals I and K. To verify these properties we invoke the results of chapters three and four and perform computer computations using "Macaulay".

Theorem 5.3. *Let I and K be the ideals in* (5.1) *and* (5.2).

(a) *I is a perfect prime ideal of* grade 5, $d(I) = 4$, $r(R/I) = 2$, *R/I is normal, and I satisfies* (CI_3).

Moreover, I is generated by a d-sequence and every ideal in the linkage class of I is strongly Cohen-Macaulay. However, $I \otimes_R \omega_{R/I}$ has nontrivial R/I-torsion, in particular, I is not strongly nonobstructed.

(b) *K is a Gorenstein prime ideal of* grade 5, $d(K) = 5$, *and R/K is normal.*

Moreover, K is strongly nonobstructed. However, K is not syzygetic, in particular, not strongly Cohen-Macaulay.

Proof. We may assume that $R = \mathbf{Q}[X]$ and $x_{56} = 0$.

We first prove part (a). Consider the R-ideals $I_0 = (f_{16}, f_{26}, f_{36}, f_{46}, f_{56})$, $I_2 = (f_{15}, f_{25}, f_{35}, f_{45}, f_{56})$, and let F be the 6 by 6 Pfaffian of X. Then I_0 and I_2 are Gorenstein ideals of grade 3 [4, 2.1] and hence I_0 is strongly nonobstructed [10, 3.3] and strongly Cohen-Macaulay [13, Example 2.2]. Using the standard Pfaffian relations it is easy to see that there is a tight double link $I_0 \sim I_1 \sim I_2$, where in the notation of Definition 1.5, $\alpha_1 = F$, $\alpha_2 = f_{56}$, $\beta = f_{46}$, $\delta = f_{45}$. Now obviously $I = I_0 + I_2$, and since I contains the regular sequence $f_{16}, f_{26}, f_{56}, f_{35}, f_{45}$, it follows that $\operatorname{grade} I \geq 5 = 3 + 2$. But then by Theorem 4.3, I is a perfect ideal of grade 5. Furthermore, a computation with "Macaulay" shows that $H_1(I)$ is Cohen-Macaulay. It is easy to see that $\nu(I) = 9$, and using the Jacobi criterion one readily checks that R/I is (R_2) and hence normal. Moreover, Remark 4.5 implies that $r(R/I) = 2$ and that $NCI(I) = NG(R/I)$. Thus to show that I satisfies (CI_3), it suffices to prove that $\operatorname{codim} NG(R/I) \geq 4$. However, since $r(R/I) = 2$, we know that $NG(R/I) = V(H/I)$ where H is the ideal generated by the entries of the last map in a minimal free R-resolution of R/I. Using "Macaulay" one easily sees that

$$H = (x_{ij} | 1 \le i < j \le 4) + I_2 \begin{pmatrix} x_{15} & x_{25} & x_{35} & x_{45} \\ x_{16} & x_{26} & x_{36} & x_{46} \end{pmatrix},$$

and hence $\operatorname{ht} H = 9$. Thus $\operatorname{codim} NG(R/I) = \operatorname{codim} V(H/I) = 4$.

We now show that every ideal in the linkage class of I is strongly Cohen-Macaulay. Since this property is preserved under even linkage [13, 1.12], it suffices to prove that I and some ideal J directly linked to I are strongly Cohen-Macaulay. Now I is a perfect prime ideal with $d(I) = 4$. Then it follows from [15, 2.13, 2.14, 2.22], that I is strongly Cohen-Macaulay once we know that $H_1(I)$ is Cohen-Macaulay. But this we had seen earlier. Moreover, $d(J) \le r(R/I) = 2$, and hence by [3], J is also strongly Cohen-Macaulay.

We know that $d(I) = 4$ and I satisfies (CI_3). Thus $\nu(I_p) \le \dim R_p$ for all $p \in V(I)$, and since moreover I is strongly Cohen-Macaulay, it follows from [16, Theorem 2.4], that I is generated by a d-sequence.

It remains to show that $I \otimes_R \omega_{R/I}$ has nontrivial R/I-torsion. We do this by using Corollary 3.10. Consider the link $J = (\alpha_1, \dots, \alpha_5) : I$, where $\alpha_1 = f_{16}$, $\alpha_2 = f_{26} - f_{56}$, $\alpha_3 = f_{36} - f_{45}$, $\alpha_4 = f_{46} + f_{45}$, $\alpha_5 = f_{15} + f_{35}$. We will prove that $\operatorname{grade}(I + J) \ge 6$, that $(I + J) \otimes_R R/I + J$ is Cohen-Macaulay, but that $H_1(I + J)$ is not Cohen-Macaulay. We first show how our claim will follow from this. Suppose that $I \otimes_R \omega_{R/I}$ is R/I-torsion free. Since $\operatorname{grade}(I+J) \ge 6$, it follows that I and J are geometrically linked, and since I and J are generically complete intersections and $H_1(I)$ and $H_1(J)$ are Cohen-Macaulay, we know that I and J are syzygetic. But then Corollary 3.10 would imply that $I + J$ is syzygetic, and therefore $H_1(I + J)$ would be a first syzygy module of the Cohen-Macaulay module $(I + J) \otimes_R R/I + J$ (cf. the discussion at the beginning of the third section). But this is impossible since $H_1(I + J)$ is not Cohen-Macaulay.

Thus to complete the proof of part (a), we have to show that $\operatorname{grade}(I+J) \ge 6$, $(I + J) \otimes_R R/I + J$ is Cohen-Macaulay, and $H_1(I + J)$ is not Cohen-Macaulay. To this end, let "′" denote reduction modulo the R-regular sequence $x_{14} - x_{45}$, x_{16}, x_{26}, $x_{35} - x_{46}$, $x_{36} - x_{45}$. We may identify R' with the polynomial ring $k[x_{12}, x_{13}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}]$. Now $\alpha_1', \dots, \alpha_5'$ still form an R'-regular sequence, and hence by Lemma 1.9, $J' = (\alpha_1', \dots, \alpha_5') : I'$. From the homogeneous minimal resolution of R'/I' and the mapping cone construction in [28], it follows that $(\alpha_1', \dots, \alpha_5') : I'/(\alpha_1', \dots, \alpha_5')$ has two homogeneous generators, one in degree 2 and one in degree 3. On the other hand, one sees from the relations on the generators of I' that the elements $h_1 = x_{23}^2 + x_{12}x_{23} - x_{24}^2$ and $h_2 = x_{25}^2 x_{34}$ are contained in $(\alpha_1', \dots, \alpha_5') : I'$. Modulo $(\alpha_1', \dots, \alpha_5')$, h_1 and h_2 minimally generate an ideal. Therefore $J' = (\alpha_1', \dots, \alpha_5') : I' = (\alpha_1', \dots, \alpha_5', h_1, h_2)$, and hence $I' + J' = (I', h_1, h_2)$. Now we have an explicit homogeneous generating set of the ideal $I' + J'$, and we are able to perform computer computations using "Macaulay". Then one easily checks that $I' + J'$

has grade 6 and is generically a complete intersection. Thus by Lemma 1.10, $(R', I'+J')$ is a specialization of $(R, I+J)$. In particular, $I+J$ has also grade 6 and is generically a complete intersection. Moreover, $I + J$ is Gorenstein by Proposition 1.3.

Write $S = R/I+J$. It remains to show that $(I+J)\otimes_R S$ is Cohen-Macaulay, whereas $H_1(I + J)$ is not Cohen-Macaulay.

To this end let

$$\begin{aligned}\overline{R} &= R'/(x_{12} - x_{34} - x_{35}, x_{15} - x_{24} - x_{34}, x_{13} - x_{25}) \\ &\cong k[x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}],\end{aligned}$$

let "$\overline{}$" denote images in $\overline{R}$, and write $\overline{S} = \overline{R}/\overline{I+J}$. Notice that $\overline{I+J} = (\overline{I}, \overline{h}_1, \overline{h}_2)$, providing us with an explicit homogeneous minimal generating set of $\overline{I+J}$ consisting of 11 elements. Using "Macaulay" one shows that $l(\overline{S}) = 25$. In particular, grade $\overline{I+J} = 6$, and hence $(\overline{R}, \overline{I+J})$ is a specialization of $(R, I + J)$. Since $I + J$ is generically a complete intersection of grade 6, it follows that the S-module $(I + J) \otimes_R S$ has a rank and that this rank is 6. Then by [9, 1.1], or [26], $(I+J)\otimes_R S$ is a Cohen-Macaulay module if and only if

$$l((I + J) \otimes_R S \otimes_S \overline{S}) = 6\ l(\overline{S}).$$

On the other hand,

$$l((I + J) \otimes_R S \otimes_S \overline{S}) = l(\overline{R}/(\overline{I+J})^2) - l(\overline{S})$$

and the above equality becomes

$$l(\overline{R}/(\overline{I+J})^2) = 7\ l(\overline{S}).$$

Now using "Macaulay" one checks that

$$l(\overline{R}/(\overline{I+J})^2) = 175 = 7\ l(\overline{S}).$$

Thus $(I + J) \otimes_R S$ is Cohen-Macaulay.

Likewise, the first Koszul homology module of a minimal generating set of $I+J$, $H_1(I+J)$, has a rank as an S-module, and this rank equals $d(I+J) = 5$. Now $H_1(I + J)$ is Cohen-Macaulay if and only if

$$l(H_1(I + J) \otimes_S \overline{S}) = 5\ l(\overline{S}).$$

On the other hand, $H_1(I + J) \otimes_S \overline{S} \cong H_1(\overline{I+J})$ by [15, 2.15], and therefore the above equality becomes

$$l(H_1(\overline{I+J})) = 5\ l(\overline{S}).$$

One can use "Macaulay" to compute the Hilbert series of $B_1(\overline{I+J})$, the module of first boundaries in the Koszul complex of a homogeneous minimal generating set of $\overline{I+J}$, and it follows that

$$l(H_1(\overline{I+J})) = 126 > 125 = 5\ l(\overline{S}).$$

Thus $H_1(I + J)$ is not Cohen-Macaulay. This finishes the proof of part (a).

We now prove part (b). Consider the R-regular sequence $f_{16}, f_{26}, f_{56}, f_{35}, f_{45}$ in I, and the link $L = (f_{16}, f_{26}, f_{56}, f_{35}, f_{45}) : I$. As in the first part of the proof, one easily computes that

$$L = (f_{16}, f_{26}, f_{56}, f_{35}, f_{45}) + x_{34}(x_{12}, x_{15}x_{26} - x_{16}x_{25}).$$

Then $L \not\subset I$ and since I is prime by part (a), it follows that I and L are geometrically linked, and hence $I + L/I \cong \omega_{R/I}$ by Proposition 1.2(c). On the other hand, since $x_{34} \notin I$ and I is prime, there is an isomorphism of R/I-modules $I+L/I \cong (I, x_{12}, x_{15}x_{26} - x_{16}x_{25})/I$, and thus the latter module is also isomorphic to $\omega_{R/I}$. Therefore $(I, x_{12}, x_{15}x_{26} - x_{16}x_{25})$ is a Gorenstein ideal of grade 6. However, from the defining equations of I and K one sees that $(I, x_{12}, x_{15}x_{26} - x_{16}x_{25}) = (K, x_{12})$, where x_{12} is a regular element on R/K. Thus K is a Gorenstein ideal of grade 5. One easily sees that $\nu(K) = 10$, and the Jacobi criterion implies that R/K is (R_1) and hence normal.

To show that K is strongly nonobstructed, but not syzygetic we proceed as in the proof of part (a). By "$\overline{}$" we denote reduction modulo the R-regular sequence $x_{12}, x_{13} - x_{24}$, $x_{14} + x_{16} - x_{34}$, $x_{14} + x_{24} + x_{25} - x_{45}$, $x_{15} - x_{26}$, $x_{23} + x_{26} + x_{45}$, x_{35}, $x_{36} - x_{45}$, x_{46}. Then $\overline{R} = k[x_{13}, x_{14}, x_{15}, x_{16}, x_{23}]$. Write $S = R/K$ and $\overline{S} = \overline{R}/\overline{K}$. Using "Macaulay" one shows that $l(\overline{S}) = 12$. In particular, $\operatorname{grade}\overline{K} = 5$ and hence $(\overline{R}, \overline{K})$ is a specialization of (R, K). Since K is generically a complete intersection, it follows as in the proof of part (a) that $K \otimes_R S$ is Cohen-Macaulay if and only if

$$l(K \otimes_R S \otimes_S \overline{S}) = 5\, l(\overline{S}),$$

which in turn is equivalent to $l(\overline{R}/\overline{K}^2) = 72$. However, using "Macaulay", we compute the Hilbert series of $\overline{R}/\overline{K}^2$ as $1 + 5t + 15t^2 + 35t^3 + 16t^4$. Thus indeed $l(\overline{R}/\overline{K}^2) = 72$, and $K \otimes_R S$ is Cohen-Macaulay. Since S is Gorenstein it now follows that K is strongly nonobstructed. From the Hilbert series of $\overline{R}/\overline{K}^2$ we also see that the dimension of the degree 4 part of $\overline{K}^2$ is 54. On the other hand, all the generators of $\overline{K}^2$ sit in degree 4, and hence $\nu(\overline{K}^2) = 54$. But then $\nu(\overline{K}^2) = 54 < 55 = \nu(S_2(\overline{K}))$, and thus $\overline{K}$ cannot be syzygetic.

Now suppose that K is syzygetic. Then there is an exact sequence

$$0 \to H_1(K) \to S^{10} \to K \otimes_R S \to 0.$$

Since $K \otimes_R S$ is Cohen-Macaulay, tensoring this sequence by $\otimes_S \overline{S}$ yields an exact sequence

$$0 \to H_1(K) \otimes_S \overline{S} \to \overline{S}^{10} \to \overline{K} \otimes_{\overline{R}} \overline{S} \to 0$$

where by [15, 2.15], $H_1(K) \otimes_S \overline{S} \cong H_1(\overline{K})$. Then the natural map $H_1(\overline{K}) \to \overline{S}^{10}$ is injective, and hence $\overline{K}$ is syzygetic, which is impossible by the above. □

In the proof of part (a) of the theorem, one could have replaced the link J of I by the link L which was also used in the proof of part (b). However, the computer computations for the latter ideal turned out to be considerably more complicated. On the other hand, if one would have only wanted to show that the module $I \otimes_R \omega_{R/I}$ is not Cohen-Macaulay, one could have directly computed the length of a zero-dimensional specialization of this module (instead of invoking Corollary 3.10). This was done by W. Vasconcelos using "Macsyma," and I am grateful for his help. I would like also to thank C. Huneke for his helpful comments concerning the material of this paper.

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